- Let us assume $\Sigma \subseteq N_{\mathbb{R}}$ is a fan such that $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_{\mathbb{R}}$

$\Rightarrow X_\Sigma$ has no torus factors

$\Updownarrow$

$\Rightarrow$ there is an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow Cl(X_\Sigma) \longrightarrow 0.$$

- **Def:** 1) The cox ring (total homogenuous cordinate ring) of a toric variety X_Σ is the $Cl(X)$-graded polynomial ring $S = k[x_\rho \mid \rho \in \Sigma(1)]$ and

$$\deg\left(\prod_\rho x_\rho^{a_\rho}\right) = \left[\sum_\rho a_\rho D_\rho\right] \in Cl(X).$$

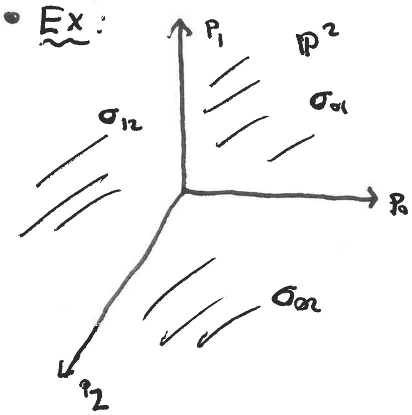
2) The irrelevant ideal of X_Σ is the ideal

$$B_\Sigma = \left\langle x^{\hat\sigma} = \prod_{\rho \notin \sigma(1)} x_\rho \;\middle|\; \sigma \in \Sigma \right\rangle$$

- **Remark:** +) The irrelevant ideal is a monomial ideal in S.

  +) $x^{\hat\tau}$ is a multiple of $x^{\hat\sigma}$ if $\tau \subseteq \sigma$ so can restrict to $\Sigma(\max)$

- **Ex:**



$$0 \to \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \xrightarrow{[1:1:1]} Cl(\mathbb{P}^2) \longrightarrow 0$$

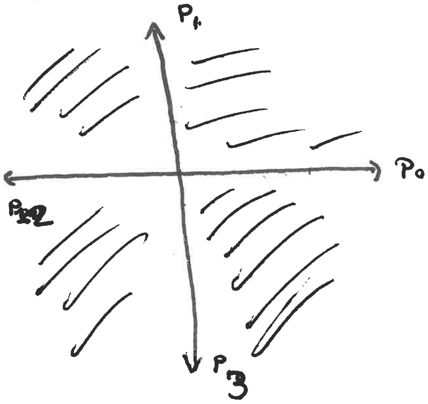
$S = \mathbb{C}[x_0, x_1, x_2]$

$\deg(x_i) = 1 \Rightarrow$ standard $\mathbb{Z}$-grading

$B = \langle \hat\sigma_{01}, \hat\sigma_{02}, \hat\sigma_{12} \rangle$

$= \langle x_2, x_1, x_0 \rangle$

• **Ex** : $\mathbb{P}^1 \times \mathbb{P}^1$



$$\begin{matrix} \binom{1}{0} & \binom{0}{1} & \binom{-1}{0} & \binom{0}{-1} \end{matrix}$$

$$S = \mathbb{C}[x_0, x_1, x_2, x_3]$$

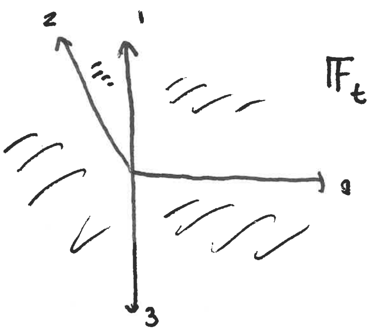
$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} Cl(\mathbb{P}^1 \times \mathbb{P}^1) \longrightarrow 0$$

$$B = \langle x_2 x_3, \; x_0 x_3, \; x_1 x_2 \; x_1 x_0 \rangle$$

$$= \langle x_0, x_2 \rangle \cap \langle x_1, x_3 \rangle$$

" transpose mingers kernel transpose "

•



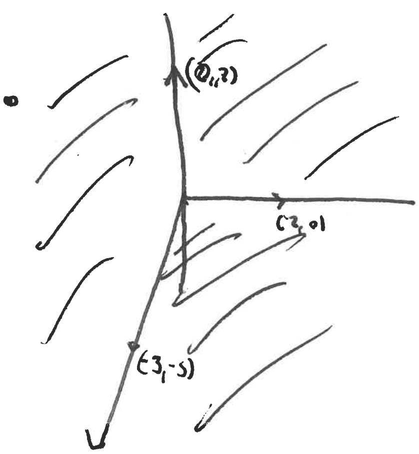
$\mathbb{F}_t$

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 5 \\ 0 & -1 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & 0 \\ -5 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}} Cl(\mathbb{F}_t) \longrightarrow 0$$

$$S = K[x_0, x_1, x_2, x_3]$$
$$\begin{matrix} \binom{1}{0} & \binom{-5}{1} & \binom{1}{0} & \binom{0}{1} \end{matrix}$$

$$B = \langle x_2 x_3, \; x_1 x_2, \; x_0 x_3, \; x_0 x_1 \rangle$$

•



$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \longrightarrow -$$

• **Thm** (Toric Null): Let X_Σ be an simplicial toric variety

① There are correspondences

$$\left\{\begin{array}{c}\text{subvarieties}\\ \text{of } X_\Sigma\end{array}\right\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \left\{\begin{array}{c}\text{radical}\\ \text{homogeneous ideals}\\ \text{in}\\ S\end{array}\right\}$$

② If $I \subseteq S$ is a homogeneous ideal then

$$\mathbb{V}(I) = \emptyset \iff B_\Sigma^{\,k} \subseteq I \quad \text{for some } k>0.$$

③ There is a bijection

$$\left\{\begin{array}{c}\text{closed subvarieties}\\ \text{of } X_\Sigma\end{array}\right\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \left\{\begin{array}{c}\text{radical homogeneous ideals}\\ I \subseteq B \subseteq S\end{array}\right\}$$

• **Thm** (Toric Serre Correspondence): Given a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules let $\mathcal{F}(\alpha) = \mathcal{O}_X(\alpha) \otimes_{\mathcal{O}_X} \mathcal{F}$ for $\alpha \in Cl(X_\Sigma)$. and define $\Gamma_*(\mathcal{F}) = \bigoplus_\alpha H^0(X, \mathcal{F}(\alpha))$

① There are correspondences

$$\left\{\begin{array}{c}\text{quasi-coherent}\\ \text{sheaves}\\ \text{on } X_\Sigma\end{array}\right\} \underset{\sim}{\overset{\Gamma_*}{\rightleftarrows}} \left\{\begin{array}{c}\text{graded modules}\\ \text{over } S\end{array}\right\}$$

② If $\mathcal{F}$ is ~~quasi~~coherent then we may take M f.g. s.t. $\widetilde{M} \cong \mathcal{F}$

③ If X is smooth then $\widetilde{M} = 0 \iff B^l M = 0$ for $l \gg 0$.

• **Definition**: The quasi-coherent sheaf $\widetilde{M}$ is defined by

$$\Gamma(U_\sigma, \widetilde{M}) = \left(M\left[\tfrac{1}{x^{\hat\sigma}}\right]\right)_0$$

- **Thm**: (Cox Construction): Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan such that $\{u_\rho\}$ spans $N_{\mathbb{R}}$. There exists a group $G \subseteq (\mathbb{C}^*)^{\Sigma(1)}$ and a subvariety $Z_\Sigma \subseteq \mathbb{A}^{\Sigma(1)}$ such that

  1) $X_\Sigma \cong (\mathbb{A}^{\Sigma(1)} \setminus Z_\Sigma) /\!/ G$ (almost geometric quotient)

  2) G-orbits of $\mathbb{A}^{\Sigma(1)} \setminus Z_\Sigma = X_\Sigma$. (geometric quotient)

- The hypothese $\Rightarrow$ we have an SES

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow Cl(X_\Sigma) \longrightarrow 0$$

Applying $Hom_{\mathbb{Z}}(-, \mathbb{C}^\times)$ and using $\mathbb{C}^\times$ is divisible

$$\begin{tikzcd}
0 \arrow[r] & Hom(Cl(X_\Sigma), \mathbb{C}^\times) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & Hom(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^\times) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & Hom(M, \mathbb{C}^\times) \arrow[r] & 0 \\
0 \arrow[r] & G \arrow[r] & (\mathbb{C}^\times)^{\Sigma(1)} \arrow[r] & T_N \arrow[r] & 0
\end{tikzcd}$$

- The subset $Z_\Sigma \subseteq \mathbb{A}^{\Sigma(1)}$ is given by $V(B_\Sigma)$ and it is a union of coordinate hyperplanes.