• Def: Let X be irreducible.

+ A prime divisor $D \subseteq X$ is an irreducible subvariety of codim $= 1$, $\dim D = \dim X - 1$

+ A Weil divisor is a formal $\mathbb{Z}$-linear sum of prime divisor

$$D = \sum a_i D_i$$

with D_i prime and $a_i \in \mathbb{Z}$ finite sum

+ We let $\mathrm{Div}(X) = \mathbb{Z}\langle \text{prime divisors} \rangle$ be the group of weil. divisors.

•

• Def: Given a prime divisor $D \subseteq X$ with X irreducible

$$\mathcal{O}_{X,D} = \{ \varphi \in k(X) \mid \underset{\text{with } U \subseteq X \text{ open}}{\varphi : U \longrightarrow k} \ \& \ U \cap D \neq \emptyset \}$$

rational functions defined on D $\equiv$ maps of D.

• Lemma: Let $U \subseteq X$ be open and affine and nonempty. If $U \cap D \neq \emptyset$ then $\mathcal{O}_{X,D} \cong \mathcal{O}_{U, U \cap D}$ given by the isomorphism $k(X) \longrightarrow k(U)$.

• Assume U is affine

$$\left\{ \begin{matrix} \text{prime divisors} \\ \text{on } U \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{height one prime} \\ \text{ideals} \\ \text{in } k[U] \end{matrix} \right\}$$

If $D = \mathbb{V}(P)$ then

$$\mathcal{O}_{X,D} = \left\{ \frac{f}{g} \in \mathrm{Frac}(k[U]) \,\middle|\, \begin{matrix} f \in k[U] \\ g \in \end{matrix} \ \& \ g \notin P \right\} \cong k[U]_P$$

• Lemma: Let R be a normal domain. If $P \subseteq R$ is a height one prime then R_P is a DVR.

• **Def**: A DVR is a domain R and function $v: \mathrm{Frac}(R)^{\times} \longrightarrow \mathbb{Z}$ with properties

such that
$$R = \{ f \in \mathrm{Frac}(R)^{\times} \mid v(f) \geq 0 \} \cup \{0\}$$

• **CoR**: If X is a normal irreducible variety $\mathcal{O}_{X,D}$ is a DVR for all prime divisors $D \subseteq X$.

Let $v_D: k(X) \longrightarrow \mathbb{Z}$ be the corresponding valuation

• **Lemma**: Let X be an irreducible normal variety. Fix $f \in k(X)^{\times}$. There are finitely many $D \subseteq X$ prime divisors such that $v_D(f) \neq 0$.

PF: Let $U \subseteq X$ be the non-empty open subset s.t. $f: U \longrightarrow k$ is a well-defined map. (If f is cts we are done). Now $V \subseteq U \subseteq X$ w/ $V = f^{-1}(k^{\times})$ is a non-empty open subset. Claim: $v_D(f) \neq 0$ then $D \subseteq X \setminus V$. Notice f is invertible on V so if $V \cap D \neq \emptyset$ then f is invertible on $V \cap D \Rightarrow f$ is a unit in $\mathcal{O}_{X,D} \cong \mathcal{O}_{V, V\cap D}$ $\Rightarrow v_D(f) = 0$. Conversely if $D \cap V = \emptyset$ then $D \subseteq X \setminus V$ but $X \setminus V$ is a subvariety dim < dim X so it has finitely many components.

• **Def**: Let X be a normal irred. variety. Given $f \in k(x)^{\times}$ the principal divisor associated to f is
$$\mathrm{div}(f) = \sum_{D} v_D(f) D \quad \in \mathrm{Div}(X).$$

• **Lemma**: The set $\mathrm{Div}_0(X) = \{\text{principal divisor}\} \subseteq \mathrm{Div}(X)$ is a subgroup

Pf: Use the valuation theory properties

$$V_D(f^{-1}) = -V_D(f) \qquad V_D(fg) = V_D(f) + V_D(g).$$

∎

- **Def**: Two Weil divisors $D, D' \in \mathrm{Div}(X)$ are *linearly equivalent* $\iff$ $D - D'$ is principal $\iff$ $D - D' = \mathrm{div}(f)$ for $f \in k(X)^\times$

  we write $D \equiv_{lin} D'$.

- **Def**: The *class group* of a normal irreducible variety X is the quotient of $\mathrm{Div}(X)$ by $\mathrm{Div}_0(X)$ equivalently $\equiv_{lin}$.

$$0 \longrightarrow \mathrm{Div}_0(X) \hookrightarrow \mathrm{Div}(X) \twoheadrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

- In general $\mathrm{Cl}(X)$ is very complicated!!

$$\mathrm{Cl}(E) \cong \underbrace{E}_{\text{elliptic curve}} \oplus \mathbb{Z}/3\mathbb{Z}$$

- **Thm**: Let Σ be a complete fan in $N_{\mathbb{R}}$. There is a short exact sequence

$$0 \longrightarrow M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

where A is the matrix $(u_1 | \cdots | u_r)^T$ where u_i is the primitive ray generator of ρ_i.

- **COR**: $\mathrm{Cl}(X_\Sigma)$ is a finitely generated abelian group.

• **Thm**: Let Σ be a fan in $N_{\mathbb{R}}$.

① There is a bijective correspondence (inclusion reversing)

$$\left\{\begin{array}{c}\text{cones } \sigma \\ \text{in } \Sigma \\ \text{codim } k\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}T_N\text{-orbits} \\ \text{in } X_\Sigma \\ \text{of dim } k\end{array}\right\}$$

$$\sigma \longmapsto O(\sigma) \cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{R}^{\times})$$

② The affine open subset

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau)$$

③ We have $\tau \preceq \sigma \Leftrightarrow O(\sigma) \subseteq O(\tau)$ and

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$$

Pf: The affine sets U_σ are T_N invariant so reduce to the affine theorem! ∎

• **Def**: The prime divisor D_ρ associated to a ray $\rho \in \Sigma(1)$ is the closure of the torus orbit $O(\rho)$.

$$D_\rho = \overline{O(\rho)}$$

By construction this is an irreducible closed T_N-invariant subset of X.

• **Def**: Let $\mathrm{Div}_{T_N}(X) \subseteq \mathrm{Div}(X)$ denote the subgroup spanned by T_N-invariant prime divisors

• **Lemma**: $\mathrm{Div}_{T_N}(X) \cong \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}\langle D_\rho \rangle$.

- Notice since $T_N \subseteq X_\Sigma$ is dense, given $m \in M$, $\chi^m : T_N \longrightarrow k^\times$ is a rational function on X_Σ so $\chi^m \in k(X_\Sigma)^\times$.

- Prop: Let X_Σ be a toric variety on a fan Σ. Let $\rho \in \Sigma(1)$ be a ray with primitive ray generator u_ρ. For any $m \in M$

$$\langle u_\rho, m \rangle = \nu_{D_\rho}(\chi^m)$$

where $\nu_{D_\rho} : k(X)^\times \longrightarrow \mathbb{Z}$ is the DVR for the divisor D_ρ.

Pf: By definition $D_\rho = \overline{O(\rho)}$ since $O(\rho) \subseteq U_\rho$ we have that $D_\rho \cap U_\rho \neq \emptyset$ and we may reduce to computing on U_ρ.

$$\nu_{D_\rho}(\chi^m) = \nu_{D_\rho \cap U_\rho}(\chi^m|_{U_\rho})$$

Since T_N is dense in U_ρ so χ^m is also a rational function on U_ρ.
As u_ρ is primitive extend to a bases $u_\rho, e_2, \dots e_n$ for N.
Then

$$U_\rho = \operatorname{Spec} k[S_\rho] = \operatorname{Spec}(k[\rho^\vee \cap M])$$

But $\rho^\vee = \{m \in M \mid \langle e_1, m \rangle \geq 0\} = \operatorname{Span}_{\mathbb{Z}}\{e_2^\vee, \dots e_n^\vee\} \oplus \mathbb{N}e_1^\vee$ and so

$$U_\rho = \operatorname{Spec} k[x_1, x_2^{\pm}, \dots x_n^{\pm}] \cong k \times (\mathbb{G}_m^{n-1})$$

Further $D_\rho \cap U_\rho = \mathbb{V}(x_1)$ we get that

$$\mathcal{O}_{U_\rho, U_\rho \cap D_\rho} = k[x_1, x_2^{\pm}, \dots x_n^{\pm}]_{\langle x_1 \rangle}$$

and $\nu_{D_\rho}(\varphi) = \ell$ where $\varphi = x_1^\ell \frac{g}{f}$ with $g, f \in k[x_1, \dots x_n]$
$\notin \langle x_1 \rangle$.

Applying this χ^m

$$\chi^m = x_1^{\langle m, e_1 \rangle} \cdots x_n^{\langle m, e_n \rangle} = x_1^{\langle u_\rho, e_1 \rangle} \underbrace{(\qquad)}_{\notin \langle x_1 \rangle}$$

∎

- **Lemma**: Given $m \in M$

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle u_\rho, m\rangle D_\rho.$$

Pf: Note χ^m is regular on $X \setminus T_N$ so for $D \subseteq X \setminus T_N$ $\nu_D(\chi^m) = 0$.

The remainder is from $X \setminus T_N = D_{\rho_1} \cup \cdots \cup D_{\rho_n}$. by orb-cone ∎

- **Prop**: Let X be a normal irreducible variety. If $U \subseteq X$ is a nonempty open subset and $D_1, \dots, D_t$ are the irred'le prime divisor components of $X \setminus U$ then there is an exact sequence

$$\bigoplus_{i=1}^{t} \mathbb{Z}\langle D_i\rangle \longrightarrow Cl(X) \longrightarrow\!\!\!\!\rightarrow Cl(U) \longrightarrow 0$$

$$\sum a_i D_i \longmapsto \sum a_i [D_i] \longmapsto \sum a_i [D_i \cap U]$$

Pf: Let $D' \subseteq U$ be a prime divisor $U \subseteq X$ dense $\Rightarrow$ $\overline{D'} \subseteq X$ is a prime divisor on X w/ $\overline{D'} \cap U = D'$. Once one checks well-defined $\Rightarrow$ surjective. Since $D_i \cap U = \emptyset$ the composition is zero.

Finally suppose $[D] \in Cl(X)$ such that $[D]|_U = 0$. i.e. $\exists f \in k(U)^\times$ with $D|_U = \mathrm{div}(f)$. but f lifts to $\tilde{f} \in k(X)^\times$ such that

$$D|_U = \mathrm{div}(f) = \mathrm{div}(\tilde{f}) \cap U = \mathrm{div}(\tilde{f})|_U$$

Hence $-\mathrm{div}(\tilde{f}) + D$ is supported on $X \setminus U \Rightarrow D - \mathrm{div}(\tilde{f}) = \mathrm{Span}(D_i)$

$\Rightarrow$ Same for classes

∎

- **Thm**: If R is a UFD then $Cl(\mathrm{Spec}(R)) = 0$.

  { height one primes in a UFD are principal.

- **Ex**:
  - $Cl(\mathbb{A}^n) = 0$
  - $Cl(T_N) = 0$

- **Thm**: Let Σ be a fan in $N_{\mathbb{R}}$. There is a ~~diagram~~ exact sequence

$$\begin{tikzcd}
M \arrow[r] & \mathrm{Div}_{T_N}(X_\Sigma) \arrow[r, two heads] & Cl(X_\Sigma) \arrow[r] & 0 \\
m \arrow[r, mapsto] & \sum\limits_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle D_\rho \arrow[r, mapsto] & \sum\limits_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle [D_\rho] &
\end{tikzcd}$$

and if $\{u_\rho\}$ span $N_{\mathbb{R}}$ the LHS is injective.

Pf:

$$\begin{tikzcd}
 & & & & Cl(T_N) = 0 \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Div}_0(X_\Sigma) \arrow[r] \arrow[dr] & \mathrm{Div}(X) \arrow[r] & Cl(X) \arrow[r] \arrow[ur] & 0 & \\
 & M \arrow[u] \arrow[r] & \mathrm{Div}_{T_N}(X) \arrow[u] \arrow[ur, two heads] & & &
\end{tikzcd}$$

Suppose $D \in \mathrm{Div}_{T_N}$ maps to 0 meaning $D = \mathrm{div}(f)$ for $f \in k(X_\Sigma)^\times$

Note $D|_{T_N} = 0 \Rightarrow \mathrm{div}(f)|_{T_N} = 0 \Rightarrow f$ is regular on $T_N \Rightarrow f|_{T_N} \in k[T_N]^\times$

$\Rightarrow f = c\chi^m$ done!

• **Def**: Let X be a normal irreducible variety. If D is a Weil divisor on X then the sheaf associated to D is

$$\mathcal{O}_X(D)(U) = \left\{ \varphi \in k(X)^{\times} \;\middle|\; (\mathrm{div}(\varphi) + D)|_U \geq 0 \right\} \cup \{0\}$$

• **Lemma**: Let $X = \mathrm{Spec}(R)$, with R - Noetherian. If D is a weil divisor on X then

$$\Gamma(X, \mathcal{O}_X(D)) = \{ \varphi \in k(X)^{\times} \mid \mathrm{div}(\varphi) + D \geq 0 \} \cup \{0\}$$

is a finitely generated R-module.

Pf: Let $D = \sum a_i D_i$. There exists $g \in R$ such that $g \in \mathbb{I}(\mathrm{supp}(D))$ and $g \neq 0$. Hence $v_{D_i}(g) > 0$, hence $\exists\, m \in \mathbb{Z}_{\geq 0}$ s.t. $m v_{D_i}(g) > a_i$ for all i.

$\Rightarrow \mathrm{div}(g^m) - D = m\,\mathrm{div}(g) - D \geq 0$. Given $f \in \Gamma(X, \mathcal{O}_X(D))$

$$\begin{aligned}
\mathrm{div}(g^m f) &= m\,\mathrm{div}(g) + \mathrm{div}(f) \\
&= m\,\mathrm{div}(g) + \mathrm{div}(f) - D + D \\
&= \underbrace{[m\,\mathrm{div}(g) - D]}_{\geq 0} + \underbrace{[\mathrm{div}(f) + D]}_{\geq 0} \quad \Rightarrow g^m f \geq 0.
\end{aligned}$$

By the lemma below $\Rightarrow g^m f \in \mathcal{O}_X(X) = R$. Hence as an R module $\Gamma(X, \mathcal{O}_X(D)) \cdot \langle g^m \rangle \subseteq R$ which is f.g. and so $\Gamma(X, \mathcal{O}_X(D))$ is f.g.

• **Lemma**: Let X be a normal variety. If $\mathrm{div}(\varphi) \geq 0$ then $\varphi \in \Gamma(X, \mathcal{O}_X)$

• **Lemma**: Let X be a normal variety and D a weil divisor on X.

Assume $X = \mathrm{Spec}(R)$. For all $f \in R \setminus \{0\}$

$$\Gamma(D(f), \mathcal{O}_X(D)) \cong \Gamma(X, \mathcal{O}_X(D))[\tfrac{1}{f}].$$

• **COR**: If X is a normal variety and D a weil divisor then $\mathcal{O}_X(D)$ is a coherent sheaf of $\mathcal{O}_X$-modules.

• **Lemma**: Let X be a normal variety.

① Let $D_1, \dots, D_t$ be distinct prime Weil divisors, and set $Y = \mathrm{Supp}(D)$ where $D = D_1 + D_2 + \cdots + D_t$.

$$\mathcal{O}_X(-D) \cong \mathcal{I}_Y$$

② Let D and D' be weil divisors. If $D \equiv_{lin} D'$ then $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$.
(converse also true)

Pf of (2): By definition $D \equiv D'$ if and only if $D = D' + \mathrm{div}(\varphi)$ for $\varphi \in k(X)^*$.
Now we have

$$f \in \Gamma(U, \mathcal{O}_X(D)) \iff [\mathrm{div}(f) + D]|_U \geq 0$$

$$\iff [\mathrm{div}(f) + D' + \mathrm{div}(\varphi)]|_U \geq 0$$

$$\iff [\mathrm{div}(f\bar{\varphi}) + D'] \geq 0 \qquad \bar{\varphi} = \varphi|_U$$

$$\iff f\bar{\varphi} \in \Gamma(U, \mathcal{O}_X(D'))$$

Hence multiplication by $\varphi|_U$ induces an isomorphism of sections.

∎

• **Prop:** Let Σ be a fan and D a T_N-invariant Weil Divisor on X_Σ

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \cong \bigoplus_{\mathrm{div}(\chi^m)+D \geq 0} \mathbb{R} \cdot \chi^m$$

Pf: Note $T_N \cap \mathrm{Supp}(D) = \varnothing$ so $D|_{T_N} = 0$. Hence if $\varphi \in \Gamma(X_\Sigma, \mathcal{O}_X(D))$ then

$\mathrm{div}(\varphi) + D \geq 0 \Rightarrow [\mathrm{div}(\varphi) + D]|_{T_N} \geq 0 = \mathrm{div}(\varphi)|_{T_N} \geq 0$. Hence

$\bar{\varphi} \in \mathcal{O}_{T_N}(T_N) = \mathbb{R}[T_N] = \mathbb{R}[M]$. Hence

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \subseteq \mathbb{R}[M].$$

The fact D is T_N-equivalent gives an T_N-action on $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ compatible with the action of T_N on $\mathbb{R}[M] \Rightarrow M$-graded.

∎

• Recall that $D = \sum a_\rho D_\rho$ is torus invariant and $\mathrm{div}(\chi^m) = \sum \langle m, u_\rho \rangle D_\rho$ so

$$D + \mathrm{div}(\varphi) = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho + \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

$$= \sum_{\rho \in \Sigma(1)} [a_\rho + \langle m, u_\rho \rangle] D_\rho$$

so $D + \mathrm{div}(\varphi) \geq 0 \iff a_\rho + \langle m, u_\rho \rangle \geq 0 \quad \forall \rho \in \Sigma(1)$

$\iff \langle m, u_\rho \rangle \geq -a_\rho \quad \forall \rho \in \Sigma(1)$

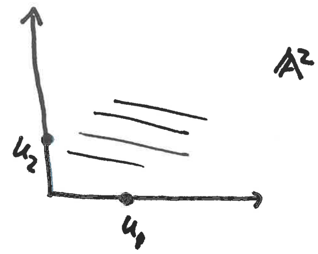
Hence define

$$P_D = \{ m \in M_\mathbb{R} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ forall } \rho \in \Sigma(1) \}$$

• **COR:** Let $D = \sum a_\rho D_\rho$ be a T_N-equivalent Weil divisor on X_Σ then

$$\Gamma(X, \mathcal{O}_X(D)) \cong \bigoplus_{m \in P_D \cap M} \mathbb{R}(\chi^m).$$

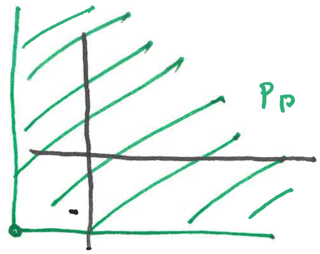
- **Ex**: $\Sigma = \mathrm{cone}(e_1, e_2)$



$D = D_1 + D_2$

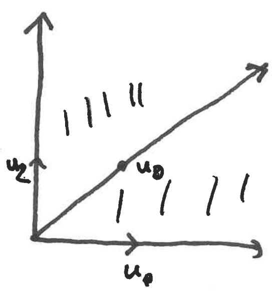
$m \in P_D \iff \langle m, u_1 \rangle \geq -1$; $\langle m, u_2 \rangle \geq -1$

$= m_1 \geq -1$; $= m_2 \geq -1$



$\Gamma(\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2}(D)) \cong k[x,y] \cdot \frac{1}{xy}$

- **Ex** $Bl_0 \mathbb{A}^2$

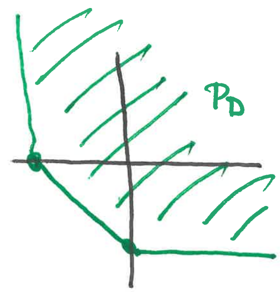


$D = D_0 + D_1 + D_2$

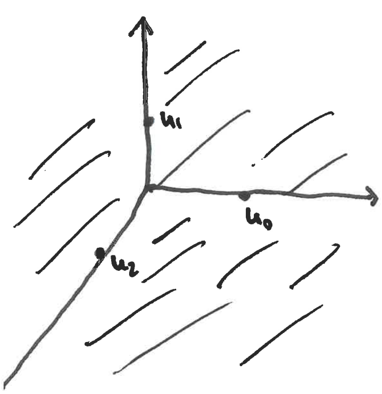
$\langle m, u_1 \rangle \geq -1 \iff m_1 \geq -1$

$\langle m, u_2 \rangle \geq -1 \iff m_2 \geq -1$

$\langle m, u_1 + u_2 \rangle \geq -1 \iff m_1 + m_2 \geq -1$



- **Ex**: $\mathbb{P}^2$

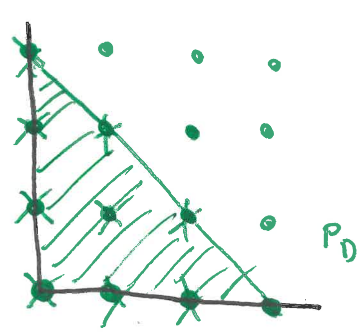


$D = 3D_2$

$\langle m, u_0 \rangle = m_1 \geq 0$

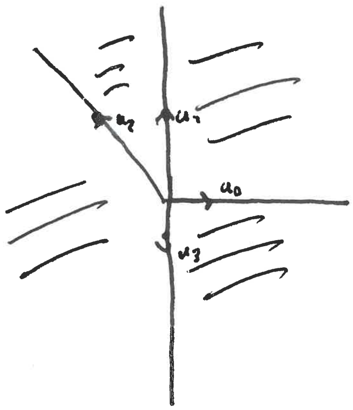
$\langle m, u_1 \rangle = m_2 \geq 0$

$\langle m, u_2 \rangle = -m_1 - m_2 \geq -3 \iff m_1 + m_2 \leq 3$



$\Gamma(\mathbb{P}^2, \mathcal{O}_D) \cong S_3$

• **Ex:** $\mathbb{F}_2$



$D = aD_2 + D_1$

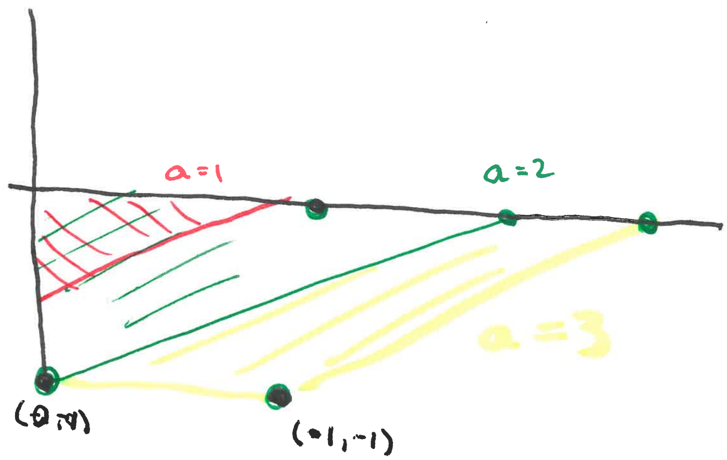
$\langle m, u_0 \rangle = m_1 \geq 0$

$\langle m, u_3 \rangle = -m_2 \geq 0 \iff m_2 \leq 0$

$\langle m, u_1 \rangle = m_2 \geq -1$

$\langle m, u_2 \rangle = -m_1 + 2m_2 \geq -a$

$\iff m_2 \geq \frac{-a}{2} + \frac{m_1}{2}$



• **Def:** If $D = \sum a_\rho D_\rho$ is a T_N-invariant divisor on X_Σ the polyhedra associated to D is

$$P_D = \{ m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1) \}$$

warning not a polytope or a lattice polytope/polyhe

• Remember: Every divisor is lin to a T_N-invariant one.

- **Lemma**: Let D and D' be T_N-invariant Weil divisors on X_Σ

① For all $k \geq 0$, $P_{kD} = kP_D$

② $P_{D+\mathrm{div}(\chi^m)} = P_D - m$

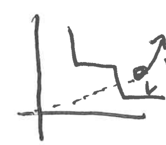
③ $P_{D+D'} = P_D + P_{D'}$

- **Prop**: Let X_Σ be a toric variety on a complete fan $\Sigma \subseteq N_{\mathbb{R}}$. For any T_N-invariant Weil divisor D

① P_D is a polytope

② $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ is a finite dim. $\mathbb{R}$-vector space.

Pf: By previous lemma ② ⟺ ①, and since P_D is defined by finitely many hyperplanes ① ⟺ P_D is bounded. Towards a contradiction assume P_D is unbounded so $\exists w \in M_{\mathbb{R}}$ ($w \neq 0$) and $v \in P_D$ s.t. $v + tw \in P_D$ $\forall t \geq 0$



this means

$$\langle v + tw, u_\rho \rangle = \langle v, u_\rho \rangle + t\langle w, u_\rho \rangle \geq -a_\rho$$

for all $\rho \in \Sigma(1)$. For this to be true for all $t \geq$ and $v \in P_D$ $\Rightarrow \langle w, u_\rho \rangle \geq 0$ for all $\rho \in \Sigma(1)$.

Since Σ is complete $|\Sigma| = N_{\mathbb{R}}$ so every $x \in N_{\mathbb{R}}$ is a positive sum of u_ρ for some ρ

$\Rightarrow \langle w, x \rangle \geq 0$ $\forall x \in N_{\mathbb{R}}$. $\Rightarrow \langle w, -x \rangle \geq 0$ $\Rightarrow \langle w, x \rangle = 0$ $\forall x \in N_{\mathbb{R}}$ $\Rightarrow w = 0$.

$\#$ ■

• **Def**: Let X be an irreducible variety. A Cartier divisor on X is represented by data $\{(U_i, f_i)\}$ where

+ $\{U_i\}_i$ an open covering of X
+ $f_i \in \Gamma(U_i, \mathcal{M}_X^\times) = k(X)^\times$

such that on $U_{ij} = U_i \cap U_j$ there exists $g_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^\times)$ such that $f_i = g_{ij} f_j$. For all i and j. Two such data are equal iff there exists a common refinement of the open covers $\{V_k\}$ such that $f_k = g_k f_k'$ for $h_k \in \Gamma(V_k, \mathcal{O}_X^\times)$ for all k.

↑ i.e. a global section of $\mathcal{M}^\times / \mathcal{O}_X^\times$.

$$\mathrm{CaDiv}(X) = \{\text{cartier divisors on } X\}$$

• **Def**: If $\{(U_k, f_k)\}$ is a Cartier divisor on an irreducible variety X then the support of $\{(U_k, f_k)\}$ is the set of points $x \in X$ such that $\exists U_k$ with $x \in U_k$ and f_k a unit in $\mathcal{O}_{X,x}$.

• Assume X is normal, let $\{(U_i, f_i)\}$ be a Cartier divisor on X. If $D \subset X$ is a prime Weil divisor define

$$\nu_D(\{U_i, f_i\}) = \nu_{D \cap U_i}(f_i)$$

For any U_i such that $U_i \cap D$ open dense non-empty. Note this is well-defined as if $U_j \cap D \neq \emptyset$ then $U_i \cap U_j \cap D \neq \emptyset$ and we get

$$\nu_{D \cap U_i}(f_i) = \nu_{D \cap U_i \cap U_j}(f_i) = \nu_{D \cap U_i \cap U_j}(g_{ij} f_j) = \nu_{D \cap U_j}(f_j).$$

$$\begin{aligned} \mathrm{CaDiv}(X) &\longrightarrow \mathrm{Div}(X) \\ \{(U_i, f_i)\} &\longmapsto \sum_{D \text{ prime}} \nu_D(\{U_i, f_i\}) D \end{aligned}$$

• Lemma: When X is normal the map $\mathrm{CoDiv}(X) \hookrightarrow \mathrm{Div}(X)$ is injective.

■

• Idea: Cartier = locally principal Weil divisor

Weil = cycle

• Lemma: If X is normal $\mathrm{Div}_0(X) \subseteq \mathrm{CoDiv}(X)$. ■

• Def: The Picard group of X is $\mathrm{Pic}(X) = \mathrm{CoDiv}(X)/\mathrm{Div}_0(X)$.

• Notice by construction $\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$ when X is normal.

• Thm: There is an exact sequence

$$M \longrightarrow \mathrm{CDiv}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0$$

and if $\{u_\rho\}$ spans $N_\mathbb{R}$ then the LHS is injective

• Thm: If Σ contains a cone of dim n then $\mathrm{Pic}(X)$ is torsion free.

• Lemma: Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex RPC.

① If D is a T_N-invariant Cartier divisor then $D = \mathrm{div}(\chi^m)$

② $\mathrm{Pic}(\mathcal{U}_\sigma) = 0$

• **Pf**: Notice ② follows from ① by the SES. Let $D = \sum a_\rho D_\rho$ be an effective a T_N-invariant cartier divisor.

$$I = \Gamma(U_\sigma, \mathcal{O}_{U_\sigma}(-D)) \subseteq k[S_\sigma]$$

is an ideal of D, and I is homogenous since D is T_N-invariant

$$I = \bigoplus_{\chi^m \in I} k\chi^m = \bigoplus_{div(\chi^m) \geq D} k \cdot \chi^m$$

Since D is Cartier it is locally principal. In paticular WLOG on U_h we have $D|_{U_h} = div(f)|_{U_h}$ for some $f \in k(U_\sigma)^\times$ and $h \in k[S_\sigma]$.

Since $div(f)|_{U_h}$ is effective we have $f|_{U_h} = \tilde{f} \in k[S_\sigma][\frac{1}{h}]$. WTS:

$$div(f) \overset{(1)}{=} \sum_\rho \nu_{D_\rho}(f) D_\rho + \sum_{E \neq D_\rho} \nu_E(f) E \overset{(2)}{\geq} \sum \nu_{D_\rho}(f) D_\rho \overset{(3)}{=} D$$

(1) is just the defintion. ~~[illegible]~~ For the remaining notice $O(\sigma) \subseteq \bigcap D_\rho$ is an open. For a point $p \in O(\sigma)$ we may pick $p \in U_h \subseteq O(\sigma)$. on which D is principal. Note $U_h \cap D_\rho \neq \emptyset$ for all ρ hence

$$\nu_{D_\rho}(f) = \nu_{D_\rho}(f|_{U_h}) = \nu_{D_\rho}(D) \qquad \forall D_\rho.$$

this gives (3) and (2) is because $f \in k[\frac{1}{h}]$. Since $f \in I$ $f = \sum c_m \chi^m$ w/ $c_m \neq 0$ and $div(\chi^m) \geq D$. $\Rightarrow$ $div(\chi^m|_U) \geq div(f|_U)$ $\Rightarrow$ $div(\chi^m/f|_U) \geq 0$

and so χ^m/f is regular on U. But

$$1 = \sum c_m \frac{\chi^m}{f}$$

...... something..... $div(\chi^m) = D$.... If D is an arbitary T_N-invariant divisor then choose $m \in \sigma^\vee \cap M$ s.t $\langle m, u_\rho \rangle > 0$, possible by SC., then For $k \gg 0$ $D' := D + div(\chi^{km})$ is effective and $D' \equiv D$.

∎

• **PF**: By the SES we want to show that if $D \in \mathrm{CoDiv}_{T_N}(X)$ and $kD \in \mathrm{div}(\chi^m)$ for some $k \gg 0$ then $D = \mathrm{div}(\chi^m)$. Assume

$$D = \sum a_\rho D_\rho \qquad \text{and} \qquad kD = \mathrm{div}(\chi^m).$$

Let σ be the n-dim cone. Then $D|_{U_\sigma}$ is Cartier and

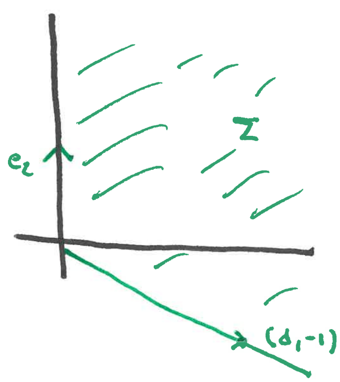
$$D|_{U_\sigma} = \sum_{\rho \in \sigma(1)} a_\rho \bar{D}_\rho = \mathrm{div}(\chi^{\bar{m}})$$

for $\bar{m} \in \sigma^\vee \cap M$. $\Rightarrow$ $a_\rho = \langle \bar{m}, u_\rho \rangle$ for $\rho \in \sigma(1)$. But $kD = \mathrm{div}(\chi^m)$ means $k a_\rho = \langle m, u_\rho \rangle$ for all $\rho \in \Sigma(1)$.

$$\langle k\bar{m}, u_\rho \rangle = k a_\rho = \langle m, u_\rho \rangle \quad \text{for all } \rho \in \sigma(1).$$

But since u_ρ span $N_{\mathbb{R}}$ this $\Rightarrow$ $k\bar{m} = m$. ∎

• **Ex**:



rational normal cone C_d

$$M \cap \sigma^\vee = \begin{cases} \langle m, (0,1) \rangle \geq 0 \\ \Leftrightarrow m_2 \geq 0 \\ \langle m, (d,-1) \rangle \geq 0 \\ \Leftrightarrow dm_1 - m_2 \geq 0 \end{cases}$$

$(1,0), (1,1), \ldots\ldots (1,d)$

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} d & -1 \\ 0 & 1 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow Cl(\hat{X}_d) \longrightarrow 0$$

$$\Rightarrow Cl(\hat{C}_d) \cong \mathbb{Z}/d\mathbb{Z}$$

$$\mathbb{R}[t_0, \ldots t_d] \twoheadrightarrow \mathbb{R}[\sigma^\vee \cap M]$$
$$t_i \longmapsto xy^i$$

$$\ker = \langle t_i t_{j+1} - t_{i+1} t_j \rangle$$

$$= \begin{pmatrix} t_0 & \cdots\cdots & t_{d-1} \\ t_1 & \cdots\cdots & t_d \end{pmatrix} \text{ 2x2 minors}$$

$$dD_1 = \mathrm{div}(\chi^{e_1}) = 0$$
$$-D_1 + D_2 = \mathrm{div}(e_2) = 0$$

$\Rightarrow$ $[D_1]$ generates

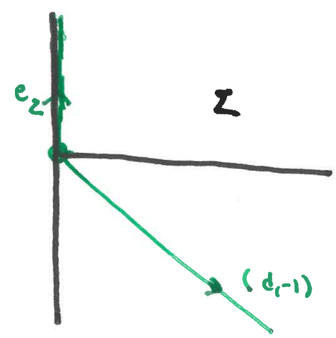
$d[D_1] = 0.$

$[D_2] = [D_1].$

• **Ex**: (Cont) Prop $\Rightarrow$ $\mathrm{Pic}(U_\sigma) = 0$

$\Rightarrow$ D_1 & D_2 are not cartier for $d > 0$

• **Ex**:



Notice X_Σ is smooth

$\Rightarrow$ $\mathrm{Cl}(X_\Sigma) = \mathrm{Pic}(X_\Sigma)$

$\cong$

$\mathbb{Z}/d\mathbb{Z}$

"class group only depends on the rays!"

$X_\Sigma = \widehat{C}_d \setminus \{*\}$

• **Prop**: Let X_Σ be a toric variety TFAE

① Every weil divisor is Cartier

② $\mathrm{Cl}(X_\Sigma) = \mathrm{Pic}(X_\Sigma)$

③ X_Σ is smooth

• **Def**: A weil divisor D is $\mathbb{Q}$-Cartier if kD is Cartier for some $k \in \mathbb{Z}_{>0}$.

• **Prop**: TFAE

① $\mathrm{Pic}(X_\Sigma) \subseteq \mathrm{Cl}(X_\Sigma)$ has finite index

② Every weil divisor is $\mathbb{Q}$-Cartier

③ X_Σ is simplicial.

• **Def**: TFAE ① X_Σ is simplicial

② Σ is simplicial

③ each cone in Σ is simplicial

④ every cone in Σ has min gens that are a basis for $N_{\mathbb{R}}$.

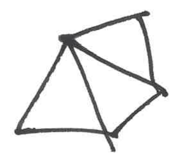
• **Thm**: Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan there is a isomorphism

$$\begin{aligned} \mathrm{CDiv}_{T_N}(X_\Sigma) &\longrightarrow SF(\Sigma, N) \\ D &\longmapsto \varphi_D \end{aligned}$$

where if $D = \sum a_\rho D_\rho$ w/ cartier data $\{m_\sigma\}$ then $\varphi_D(u) = \langle m_\sigma, u \rangle$ for $u \in \sigma$.

• **Def**: A function $\varphi : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ is convex iff $\forall t \in [0,1]$ and $u, v \in N_{\mathbb{R}}$

$$\varphi(tu + (1-t)v) \geq t\varphi(u) + (1-t)\varphi(v)$$



• **Lemma**: Let $|\Sigma|$ be full dim w/ convex support and $D = \sum a_\rho D_\rho$ a cartier divisor w/ cartier data info $\{m_\sigma\}$. TFAE

1. D is base point free
2. $m_\sigma \in P_D \ \forall \sigma \in \Sigma(n)$
3. φ_D is convex

If Σ is complete we may add

4. $P_D = \mathrm{conv}(m_\sigma \mid \sigma \in \Sigma(n))$
5. $\{m_\sigma\}$ vertices of P_D.

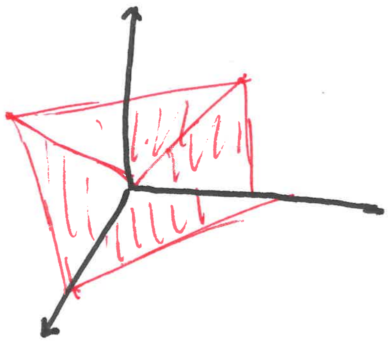
• **Def**: φ_D is strictly convex $\Leftrightarrow$ it is convex and $\varphi_D(u) = \langle m_\sigma, u \rangle \Leftrightarrow u \in \sigma$

• **Thm**: Let Σ be complete

$$D \text{ is ample} \iff \varphi_D \text{ is strictly convex}$$

and if $n \geq 2$ and D is ample then kD is very ample $k \geq n-1$.

• **Ex** $\sigma = \text{cone}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ $\sigma^\vee = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$



$$\frac{\mathbb{R}[x_1, x_2, x_3, x_4]}{\langle x_1x_2 - x_3x_4}$$

• **Thm** : Let $D = \sum a_\rho D_\rho$ on a toric variety X_Σ. TFAE

① D is Cartier

② D is principal on U_σ for all $\sigma \in \Sigma$

③ For each $\sigma \in \Sigma$ there exists $m_\sigma \in M$ w/ $\langle m_\sigma, u_\rho \rangle = -a_\rho$ $\forall \rho \in \sigma$

④ For each $\sigma \in \Sigma_{max}$ ____________________

Further if D is cartier m_σ is unique modulo $\sigma^\perp \cap M = M(\sigma)$ and if τ is a face of σ then $m_\sigma \equiv m_\tau / M(\tau)$.

• **Def** : Let Σ be a fan in $N_{\mathbb{R}}$

1) A support function on Σ is a function $\varphi : |\Sigma| \longrightarrow \mathbb{R}$ s.t. on each cone φ is linear.

2) A support function $\varphi : |\Sigma| \longrightarrow \mathbb{R}$ is integral w/r/t N if and only if $\varphi(|\Sigma| \cap N) \subseteq \mathbb{Z}$

$SF(\Sigma) = \{\text{support function}\}$ $SF(\Sigma, N) = \{\text{support function w/ int}\}$