- **Recall**: A projective variety is a quasiprojective variety V isomorphic to a projective algebraic set

$$W = \mathbb{V}(F_1, \dots F_t) = \{\bar{a} \in \mathbb{P}^n_k \mid F_i(a) = 0 \ \forall i\}$$

for some n where $F_1, \dots F_t$ are homogenous of degree d. in $k[x_0, \dots, x_n]$.

A quasiprojective variety is a locally closed subset of $\mathbb{P}^n_k$ in the Zariski topology.

- **Def**: If V is a quasiprojective variety. and $U \subseteq V$ open.
  - For $p \in U$ we say a function $\varphi : U \longrightarrow k$ is regular at p iff there exists an affine open subset $U' \subseteq U$ such that $\varphi|_{U'}$ is regular at p.
  - The function $\varphi : U \longrightarrow k$ is regular on U iff it is regular at every point $p \in U$.

  Recall if U is affine $\varphi : U \longrightarrow k$ is regular at p iff $\exists f, g \in k[U]$ s.t. $g(p) \neq 0$ and $\varphi|_{U'} = \frac{f}{g}$ on some open $p \in U' \subseteq U$.

- **Def**: ~~An~~ abstract algebraic variety is a topological space V and a sheaf of k-algebras $\mathcal{O}_V$ such that there is an open cover $V = \bigcup_i U_i$ where $(U_i, \mathcal{O}_V|_{U_i}) \cong (W_i, \mathcal{O}_{W_i})$ for W_i affine.

• **Lemma**: Suppose $\{V_\alpha\}_\alpha$ is a set of affine varieties, $\{V_{\alpha\beta}\}$ is a collection of Zariski open sets $V_{\beta\alpha} \subseteq V_\alpha$, and isomorphisms $\{V_{\beta\alpha} \xrightarrow{g_{\beta\alpha}} V_{\alpha\beta}\}$. If we have

1) $g_{\alpha\alpha} = \mathrm{Id}_{V_\alpha}$ for all α

2) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ for all α, β

3) $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$ and $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on $V_{\beta\alpha} \cap V_{\gamma\alpha}$.

then there exists an abstract algebraic X w/ an open affine cover $\{U_\alpha\}$ such that $U_\alpha \cong V_\alpha$.

• **Pf**: Define an equivalence relation $\sim$ on $\coprod_\alpha V_\alpha$ by $x \sim y$ if and only if $x \in V_\alpha$ and $y \in V_\beta$ and $g_{\beta\alpha}(x) = y$ for some α, β. Conditions 1, and 2 $\Rightarrow$ reflexivity and symmetry of $\sim$. 3 $\Rightarrow$ transitivity of $\sim$. Hence $X = \coprod_\alpha V_\alpha / \sim$ w/ quotient topology giving cts maps

$$\begin{tikzcd} V_\alpha \arrow[r, hook, "i"] \arrow[rr, bend left, "\psi_\alpha"] & \coprod V_\alpha \arrow[r, two heads, "\pi"'] & X. \end{tikzcd}$$

For each α let U_α be the image of V_α along these maps

$$U_\alpha = \{[a] \in X \mid a \in V_\alpha\}$$

Note 1) implies the composition is injective. To see it is open use that ψ is open use that

$$\pi^{-1}(\psi(V_\alpha)) \cap V_\beta = \cancel{g_{\alpha\beta}^{-1}(V_{\alpha\beta})}$$

you do the rest.

• **Def**: Let N be a lattice, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. A **fan** is a finite collection $\Sigma = \{\sigma\}$ of strongly convex rational polyhedral cones $\sigma \subseteq N_{\mathbb{R}}$ such that:

① For all $\sigma \in \Sigma$ every face of σ is in Σ

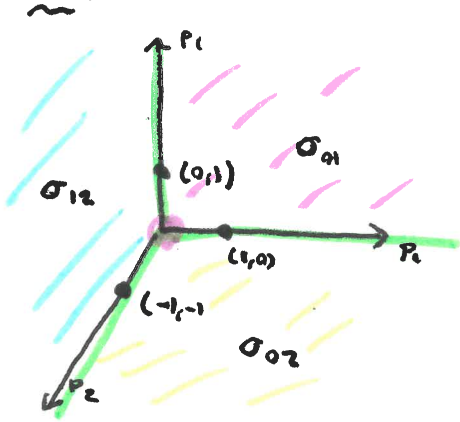
② For all $\sigma_1, \sigma_2 \in \Sigma$ the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

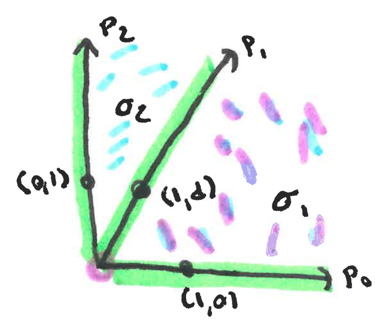
• Some consequences: (assume $\Sigma \neq \emptyset$):

+) If $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2 \in \Sigma$.

+) $0 \in \Sigma$

• **Ex**:





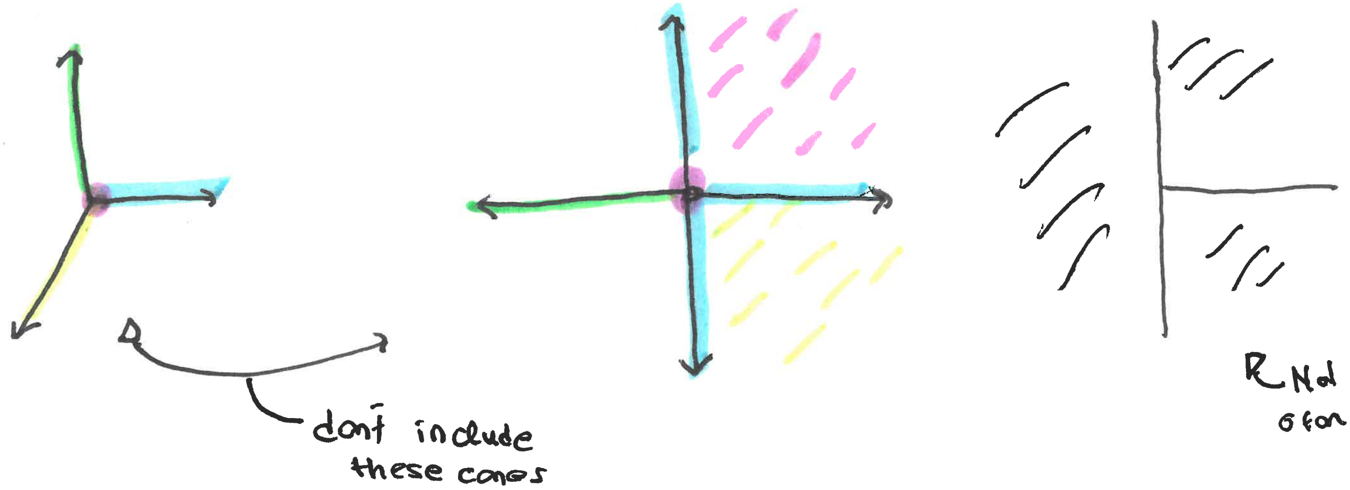
$\Sigma = \{0, \rho_0, \rho_1, \rho_2, \sigma_1, \sigma_2\}$

$\Sigma = \{0, \rho_0, \rho_1, \rho_2, \sigma_{01}, \sigma_{12}, \sigma_{02}\}$

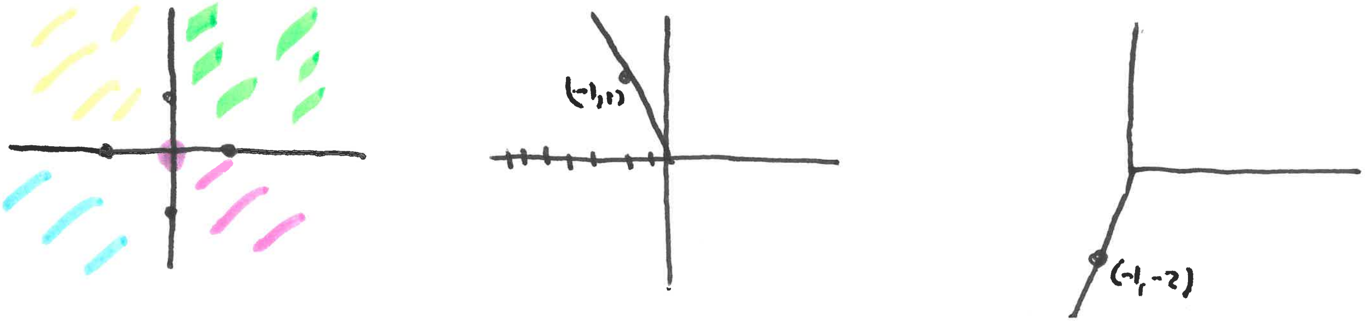
• **Lemma**: A fan is determined by its top dimensional cones.

↳ Not by its rays

• **Ex**



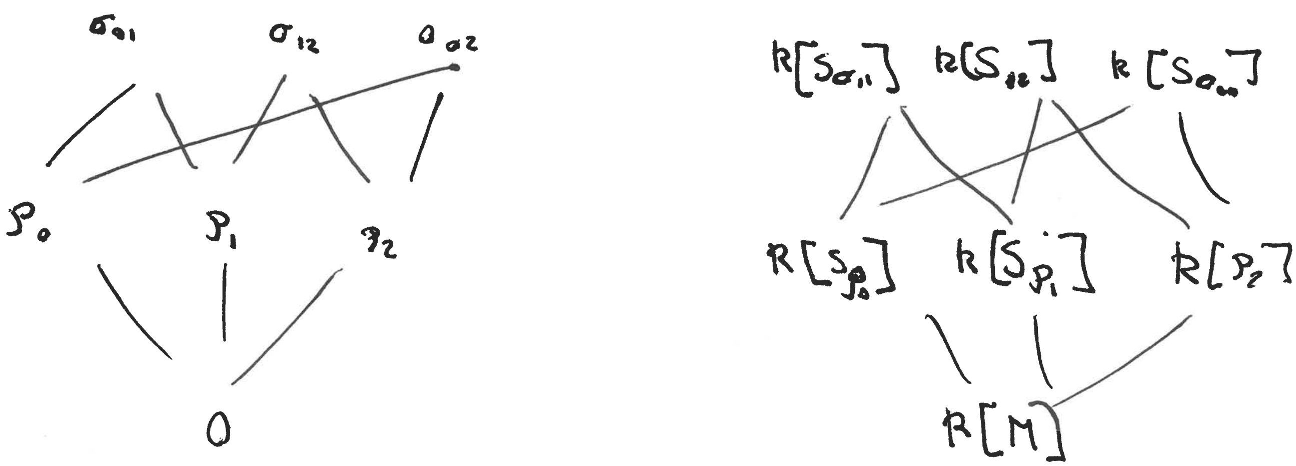
•



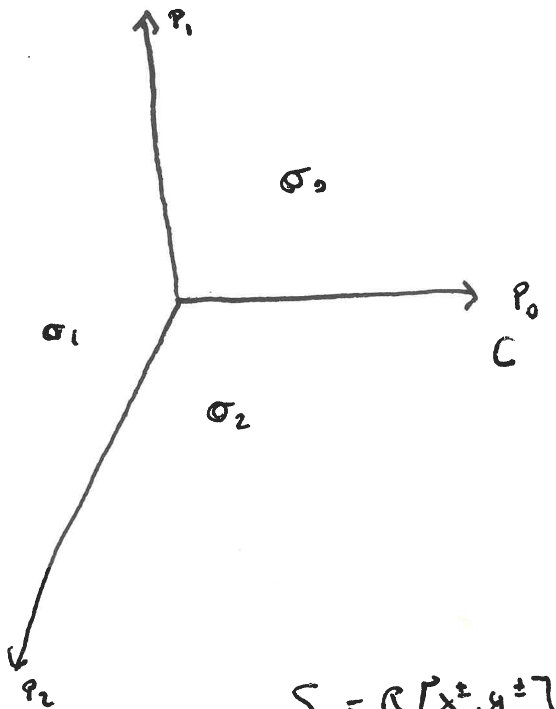
• **Def**: The support of a fan is $|\Sigma| = \bigcup \sigma \subseteq N_{\mathbb{R}}$

• For $r \in \mathbb{Z}_{\geq 0}$ we let $\Sigma(r) = \{r\text{-dim cones in } \Sigma\}$.

• **Note** we can form a poset from a fan (face poset)



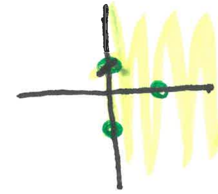
• <u>Ex</u>



$\sigma^\vee \cap M = M$

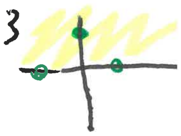
$\rho_0^\vee \cap M = \{ m \mid \langle m, (1,0) \rangle \geq 0 \}$

$= \{ (m_1, m_2) \mid m_1 \geq 0 \}$



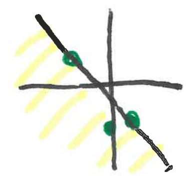
$\rho_1^\vee \cap M = \{ (m_1, m_2) \mid \langle m, (0,1) \rangle \geq 0 \}$

$= \{ (m_1, m_2) \mid m_2 \geq 0 \}$



$\rho_2^\vee \cap M = \{ m \mid \langle m, (-1,-1) \rangle \geq 0 \}$

$= \{ (m_1, m_2) \mid m_1 + m_2 \leq 0 \}$

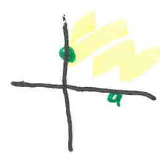


$S_\sigma = \mathbb{C}[x^{\pm}, y^{\pm}]$

$S_{\rho_0} = k[x, y, y^{-1}]$ $\quad S_{\rho_1} = k[x, x^{-1}, y]$ $\quad S_{\rho_2} = k[xy^{-1}, yx^{-1}, y^{-1}]$

$\sigma_0^\vee \cap M = \{ m \mid \langle m, (1,0) \rangle, \langle m, (0,1) \rangle \geq 0 \}$

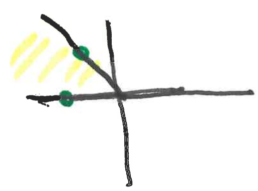
$= \{ (m_1, m_2) \mid m_1, m_2 \geq 0 \}$



$S_{\sigma_0} = k[x, y]$

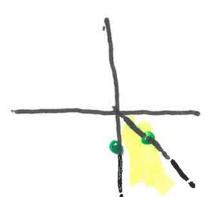
$\sigma_1^\vee \cap M = \{ m \mid \langle m, (0,1) \rangle, \langle m, (-1,-1) \rangle \geq 0 \}$

$= \{ (m_1, m_2) \mid m_2 \geq 0, \ m_1 + m_2 \leq 0 \}$



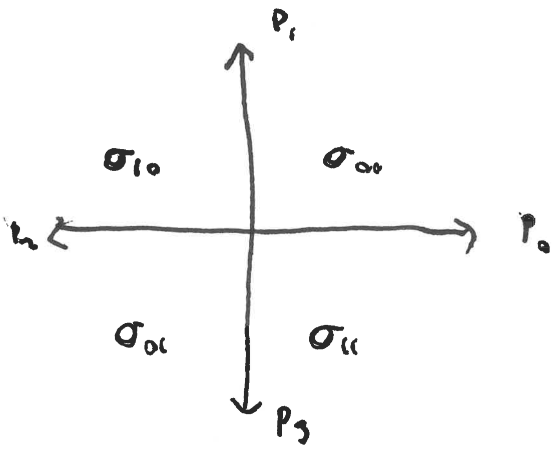
$S_{\sigma_1} = k[x^{-1}, x^{-1}y]$

$\sigma_2^\vee \cap M = \{ (m_1, m_2) \mid m_1 \geq 0 \quad m_1 + m_2 \leq 0 \}$



$S_{\sigma_2} = k[y^{-1}, xy^{-1}]$.

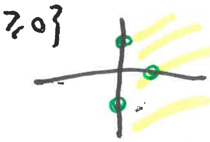
• **Ex**: $\mathbb{P}^1 \times \mathbb{P}^1$



$0^\vee \cap M = M$

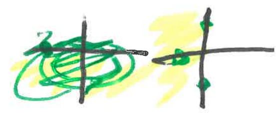
$\rho_0^\vee \cap M = \{(m_1, m_2) \mid \langle m, (1,0)\rangle \geq 0\}$

$= \{(m_1, m_2) \mid m_1 \geq 0\}$

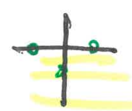


$\rho_1^\vee \cap M = \{(m_1, m_2) \mid m_2 \geq 0\}$

$\rho_2^\vee \cap M = \{(m_1, m_2) \mid m_1 \leq 0\}$



$\rho_3^\vee \cap M = \{m_2 \leq 0\}$



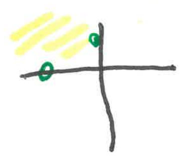
$S_0 = k[x^\pm, y^\pm]$, $S_{\rho_0} = k[x, y, y^{-1}]$ $S_{\rho_1} = k[y, x, x^{-1}]$

$S_{\rho_2} = k[x^{-1}, y^{-1}, y]$ $S_{\rho_3} = k[x^{-1}, x, y^{-1}]$

$\sigma_{00}^\vee \cap M = \{m_1 \geq 0, m_2 \geq 0\}$

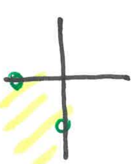
$k[x, y]$

$\sigma_{10}^\vee \cap M = \{m_1 \leq 0, m_2 \geq 0\}$



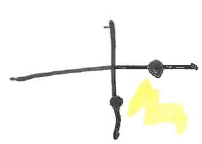
$k[x^{-1}, y]$

$\sigma_{01}^\vee \cap M = \{m_1 \leq 0, m_2 \leq 0\}$



$k[x^{-1}, y^{-1}]$

$\sigma_{11}^\vee \cap M = \{m_1 \geq 0, m_2 \leq 0\}$



$k[x, y^{-1}]$