• **Def**: Let R be an integral domain. An element $\lambda \in K = \mathrm{Frac}(R)$ is integral over R iff there exists a monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x]$$

such that $f(\lambda) = 0$. A ring R is integrally closed iff given $\lambda \in K$ such that λ is integral then $\lambda \in R$. ← also called normal.

• **Def**: An irred. affine variety V is normal $\Leftrightarrow$ $k[V]$ is integrally closed.

• **Def**: The integral closure of a domain R is the set

$$\overline{R} = \{\lambda \in \mathrm{Frac}(R) \mid \lambda \text{ integral over } R\}$$

← R is normal iff $R = \overline{R}$

• Don't confuse w/ algebraic closure.

• Notice there is an inclusion
$R \xhookrightarrow{\nu^\#} \overline{R}$ since given $r \in R$
r satisfies $(x - r) = 0$

• **Def**: Let V be an irreducible affine variety the normalization of V is the affine variety $\overline{V}$ and the dominant morphism $\nu : \overline{V} \longrightarrow V$ induced by $k[V] \xhookrightarrow{\nu^\#} \overline{k[V]}$.

• **Ex**: Let $f/g \in k(X) = \mathrm{Frac}(k[x])$ where $f, g \in k[x]$, $g \neq 0$, and $\gcd(f,g) = 1$
Assume $\exists$ $x^n + a_{n-1}x^{n-1} + \cdots a_1x + a_0 \in k[x][x]$ such that

$$\left(\frac{f}{g}\right)^n + a_{n-1}\left(\frac{f}{g}\right)^{n-1} + \cdots + a_1\left(\frac{f}{g}\right) + a_0 = 0$$

$$\Rightarrow f^n = -g(a_{n-1}f^{n-1} + \cdots + a_0 g^{n-1})$$

This means $g \mid f^n$ but since $k[x]$ is a UFD $\Rightarrow$ $g \in k^\times$ $\Rightarrow$ $f/g \in k[x]$.

• **Lemma**: If R is a UFD then R is integrally closed. ■

• **Ex**: $\mathbb{A}^1$ and $\mathbb{A}^n$ are normal.

• ~~Lemma~~

Ex: Let $V = \mathbb{V}(x^3 - y^2) \subseteq \mathbb{A}^2$.

$$k[V] = \frac{k[x,y]}{\langle x^3 - y^2 \rangle}$$

Now consider $\bar{y}/\bar{x} \in k(V)$, since $\bar{x} \in \langle x^3 - y^2 \rangle$ this is a well-defined element. Consider $t^2 - \bar{x} \in k[V][t]$

$$(\bar{y}/\bar{x})^2 - \bar{x} = \bar{y}^2/\bar{x}^2 - \bar{x} = \bar{x}^3/\bar{x}^2 - \bar{x}^2 = 0.$$

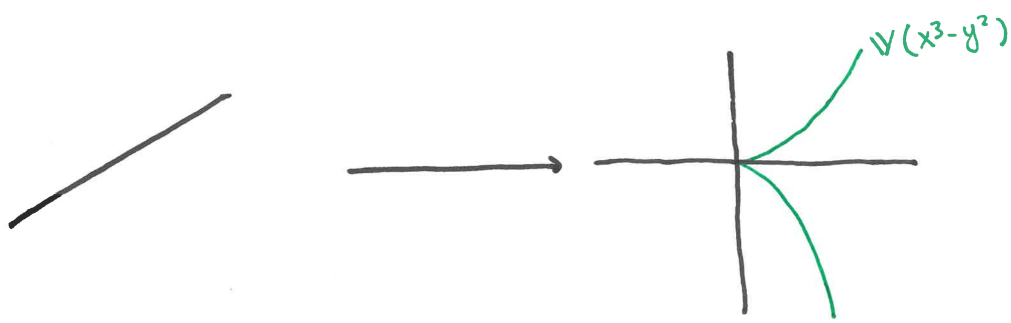
So $\bar{y}/\bar{x}$ is integral over $k[V]$. Now $\bar{y}/\bar{x} \in k[V]$ iff $\bar{x}$ divides $\bar{y}$ i.e. $\bar{y} \in \langle \bar{x} \rangle$ equivalently $y \in \langle x, x^3 - y^2 \rangle$. If this were true

$$y = x f(x,y) + (x^3 - y^2) g(x,y)$$

i.e. letting $x = 0$, $y = -y^2 g(0,y)$, which is impossible. So $k[V]$ is not integrally closed.

Using $k[V] \cong k[t^2, t^3]$ we get $\mathrm{Frac}(k(V)) = k(t)$. The isomorphism gives $t = \bar{y}/\bar{x}$ so $\overline{k[V]} \supseteq k[t]$ and converse is clear.

$$k[t^2, t^3] \hookrightarrow k[t] \qquad \text{Normalization}$$



• **Def**: Let S be an affine semigroup. The <u>saturation</u> of S is the semigroup $S^{Sat} \subseteq \mathbb{Z}S$:

$$S^{Sat} = \left\{ v \in \mathbb{Z}S \;\middle|\; mv \in S \text{ for } m \geq 1 \right\}$$

• Notice $S \subseteq S^{Sat}$ since we may take $m = 1$.

$$S \subseteq S^{Sat} \subseteq \mathbb{Z}S$$

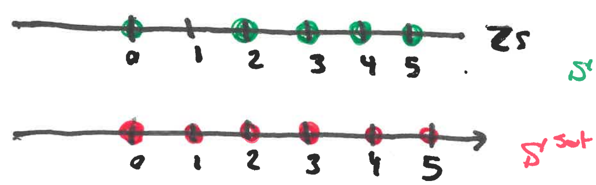
$$\Rightarrow k[S] \subseteq k[S^{Sat}] \subseteq k[\mathbb{Z}S]$$

$$\mathrm{Frac}(k[S]) \cong \mathrm{Frac}(k[S^{Sat}]) \cong \mathrm{Frac}(k[\mathbb{Z}S]).$$

• **Ex**: $S = \mathbb{N}\{2,3\}$, $\mathbb{Z}S = \mathbb{Z}$

$S^{Sat} = \mathbb{N}$

$2 \cdot 1 \in S \Rightarrow 1 \in S^{Sat}$!



• **Lemma**: Let R be a domain on $K = \mathrm{Frac}(R)$. If $\varphi: K \to K$ is a field automorphism s.t. $\varphi(R) \subseteq R$ then $\varphi(\overline{R}) \subseteq \overline{R}$.

Pf: Let $b \in \overline{R}$, then $b^n + a_{n-1}b^{n-1} + \cdots a_1 b + a_0 \overset{=0}{} \in R$ and so

$$\varphi(b^n + \cdots + a_1 b + a_0) = \varphi(b)^n + \varphi(a_{n-1})\varphi(b)^{n-1} + \cdots \varphi(a_1)\varphi(b) + \varphi(a_0)$$

$$= \varphi(0) \in R.$$

But $\varphi(R) = R \Rightarrow \varphi(a_i) \in R$ so $\varphi(b) \in \overline{R}$. Hence $\varphi(B) \subseteq B$. The same argument applied to $\varphi^{-1}: K \to K$ shows $\varphi^{-1}(B) \subseteq B$. ∎

• **Lemma:** Let S be an affine semigroup, $M = \mathbb{Z}S$, and $N = M^\vee$. If S is saturated then $k[S]$ is integrally closed.

Pf: Let $f \in \mathrm{Frac}(k[S])$ be integral over $k[S]$ so

$$f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 = 0 \qquad (*)$$

where the $a_i \in k[S]$. Since $k[S] \subseteq k[\mathbb{Z}S]$ this equation shows f is integral over $k[\mathbb{Z}S]$ - recall $\mathrm{Frac}(k[S]) \cong \mathrm{Frac}(k[\mathbb{Z}S])$ so this make sense. But $k[\mathbb{Z}S]$ is a UFD $\Rightarrow$ integrally closed and so $f \in k[\mathbb{Z}S]$. Hence

$$f = c_1 \chi^{s_1} + \cdots\cdots c_r \chi^{s_r} \qquad \text{for } c_i \in k^\times$$

and $s_i \in \mathbb{Z}S$. We must show $s_i \in S$.

First consider the case $f = \chi^s$. Express the $a_i = \sum a_{i,t}\chi^t$ for $t \in S$ and $a_{i,t} \in k^\times$. Then $(*)$ becomes

$$\chi^{ns} + a_{n-1}\chi^{(n-1)s} + \cdots + a_1\chi^s + a_0$$

$$= \chi^{ns} + \left(\sum a_{n-1,t}\chi^t\right)\chi^{(n-1)s} + \cdots + \left(\sum a_{1,t}\chi^t\right)\chi^s + \sum a_{0,t}\chi^t = 0$$

as an equation in $k[\mathbb{Z}S]$. But the characters are linearly independent $/k$. Hence χ^{ns} must cancel another term, i.e.

$$ns = t + (n-j)s \qquad \text{for } t \in S \quad j \in 0, \ldots n-1$$

$$\Rightarrow ns - (n-j)s = t$$

$$\Rightarrow js = t \in S \qquad \Rightarrow ts \in S \Rightarrow s \in S \text{ since } S \text{ is saturated}$$

• Pf: (cont.): Now consider a general f. Let

$$\mathrm{Supp}(f) = \{s_1, \ldots, s_t\} \subseteq \mathbb{Z}S.$$

Since the s_i are distinct and $\mathbb{Z}S \cong \mathbb{Z}^t$ we may choose a group homo.

$$\mathbb{Z}S \xrightarrow{\ell} \mathbb{Z}$$

such that $\ell(s_i) \neq \ell(s_j)$.

• Lemma: Given finitely many non-zero vectors $v_1, \ldots v_t$ in $\mathbb{Z}^d$ there exists a group hom $\ell: \mathbb{Z}^d \longrightarrow \mathbb{Z}$ such that $\ell(v_i) \neq 0 \;\; \forall i = 1, \ldots, t$.

Pf: For an integer N define $\ell_N: \mathbb{Z}^d \longrightarrow \mathbb{Z}$ by

$$\ell_N(x_1, \ldots x_d) = x_1 + N x_2 + \cdots + N^{d-1} x_d \quad \text{"}(1, N, \ldots N^{d-1}) \cdot v\text{"}$$

this is a linear function. Now

$$\ell_N(v_1), \ldots\ldots \ell_N(v_t)$$

are non-zero polynomials in the variable N! Since polynomials in one variable have finitely many roots for each v_i there are finitely many N s.t. $\ell_N(v_i) = 0$. Choose N away from this finite set.

■

To get our ℓ apply the lemma to all pairwise differences $s_i - s_j$.

For $a \in k^*$ define an automorphism $\tau_a: k[\mathbb{Z}S] \longrightarrow k[\mathbb{Z}S]$ by $\tau_a(x^m) = a^{\ell(m)} x^m$ and extending linearly. Note

$$\tau_a(x^m x^{m'}) = \tau_a(x^{m+m'}) = a^{\ell(m+m')} x^{m+m'} = a^{\ell(m)+\ell(m')} x^m x^{m'}$$

$$= \tau_a(x^m)\, \tau_a(x^{m'})$$

Pf: (cont.): Hence τ_a is a k-algebra homomorphism. Note $\tau_{a^{-1}}$ is the inverse of τ_a so τ_a is a k-algebra homo. By previous lemma $\Rightarrow$ $\tau_a(f)$ is integral over $k[S]$ for all $a \in k^\times$. Say.

$$f = c_1 x^{m_1} + \cdots \quad c_r x^{m_r}$$

Then consider $\tau_{a^i}(f)$ for $i = 0, 1, \ldots r-1$

$$\tau_1(f) = c_1 x^{m_1} + \cdots \quad c_r x^{m_r}$$

$$\tau_a(f) = c_1 a^{\ell(m_1)} x^{m_1} + \cdots + c_r a^{\ell(m_r)} x^{m_r}$$

$$\tau_{a^2}(f) = c_1 a^{2\ell(m_1)} x^{m_1} + \cdots \quad c_r a^{2\ell(m_r)} x^{m_r}$$

$$\vdots$$

$$\tau_{a^{r-1}}(f) = c_1 a^{(r-1)\ell(m)} x^{m_1} + \cdots \quad c_r a^{(r-1)\ell(m_r)} x^{m_r}$$

That is letting $\lambda_i = a^{\ell(m_i)}$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \vdots \\ \lambda_1^2 & \lambda_2^2 & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \cdots & \lambda_r^{r-1} \end{pmatrix} \begin{pmatrix} c_1 x^{m_1} \\ c_2 x^{m_2} \\ \vdots \\ c_r x^{m_r} \end{pmatrix} = \begin{pmatrix} f \\ \tau_a(f) \\ \vdots \\ \tau_{a^{r-1}}(f) \end{pmatrix}$$

Since the λ_i are distinct the matrix on the LHS is invertible since it is Vandermondt w/ det $= \prod_{i \neq j} (\lambda_i - \lambda_j)$, as k is infinite and so we may choose the a_i's appropriately.

Now this means $c_i x^{m_i}$ is a k-linear combination of integral elements $\Rightarrow$ it is integral over $k[S]$. The monomial case $\Rightarrow$ $x^{m_i} \in S$ $\Rightarrow$ $f \in k[S]$ as needed.

$\blacksquare$

• **Lemma**: Let S be an affine semigroup. If $k[S]$ is integrally closed then S is saturated.

• **Pf**: Let $v \in \mathbb{Z}S$ such that $mv \in S$ with $m \geq 1$. Consider the polynomial

$t^m - \chi^{mv}$ this is monic w/ coefficients in $k[S]$ and $(\chi^v)^m - \chi^{mv} = 0$.

Hence χ^v is integral over $k[S] \Rightarrow \chi^v \in k[S] \Rightarrow v \in S$.

$\blacksquare$

• **Cor**: Let S be an affine semigroup.

$$k[S] \text{ is normal } \Leftrightarrow S \text{ is saturated.}$$

• **Prop**: Let S be an affine semigroup. The normalization of X_S is induced by the map

$$k[S] \xhookrightarrow{i^*} k[S^{sat}].$$

Pf: By construction $S \subseteq S^{sat}$ so the inclusion i^* exists. Let $f \in \overline{k[S]}$. then a

$$f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 = 0 \qquad \text{for } a_i \in k[S]$$

but then $a_i \in k[S^{sat}] \Rightarrow f$ is integral over $k[S^{sat}] \Rightarrow f \in k[S^{sat}]$

Since saturated $\Rightarrow$ integrally closed. Hence

$$\overline{k[S]} \subseteq k[S^{sat}].$$

Now let $\chi^v \in k[S^{sat}]$ then $mv \in S$ for $m \geq 1$ and so χ^v satisfies $t^m - \chi^{mv} \in k[S][t]$. Hence χ^v is integral over $k[S] \Rightarrow \chi^v \in \overline{k[S]}$

But since $\overline{k[S]}$ is a ring and χ^v span $k[S^{sat}] \Rightarrow$

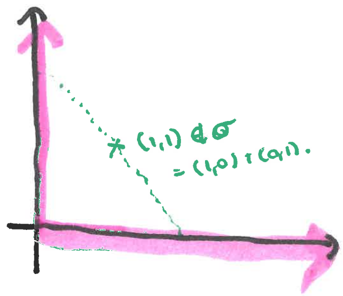
$$k[S^{sat}] \subseteq \overline{k[S]}.$$

$\blacksquare$

- **Def**: Let N be a lattice $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

  - A cone $\sigma \subseteq N_{\mathbb{R}}$ is a subset such that $x \in \sigma \Rightarrow \lambda x \in \sigma \ \forall \lambda \geq 0$.

  - A cone σ is a convex cone iff $\lambda_1 x + \lambda_2 y \in \sigma \ \forall x, y \in \sigma$ and $\lambda_i \geq 0$.



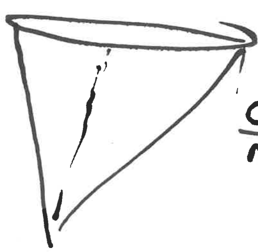
Cone since
$\lambda(1,0) = (\lambda, 0)$ and $\lambda(0,1) = (0,\lambda) \in \sigma$
But not convex.

  - Given a set $S \subseteq N_{\mathbb{R}}$ the cone generated by S is

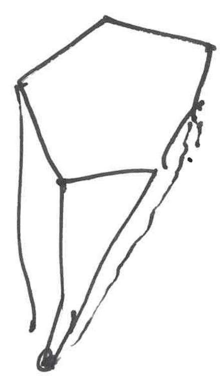
$$\mathrm{cone}(S) = \left\{ \sum_{u \in S} \lambda_u u \;\middle|\; \lambda_u \geq 0 \right\} \subseteq N_{\mathbb{R}}.$$

Note by definition this is convex!

  - A cone $\sigma \subseteq N_{\mathbb{R}}$ is polyhedral iff $\sigma = \mathrm{cone}(S)$ for some finite subset $S \subseteq N_{\mathbb{R}}$



Convex
not polyhedr



- A cone σ is rational if $\sigma = \mathrm{cone}(S)$ for $S \subseteq N$.

- **Lemma**: Let $S \subseteq M$ be an affine semigroup in lattice M

$$S^{sat} = \mathrm{cone}(S) \cap M \subseteq M_{\mathbb{R}}.$$

• **Lemma**: (Gordon's Lemma): If $\sigma \subseteq N_{\mathbb{R}}$ is a rational polyhedral cone then $\sigma \cap N$ is a f.g. semigroup

$\uparrow$ Since σ is RPC $\Leftrightarrow$ $\sigma = \text{cone}(S)$ for $S \subseteq N$ finite and $\sigma \cap N = S^{sat}$ Gordon's lemma is really

S' f.g. semi-group $\Rightarrow$ S'^{sat} f.g. semigroup.

• **Prop**: There is a bijection, for a fixed lattice N.

$$\left\{ \begin{array}{c} \text{finitely generated saturated} \\ \text{affine semigroups} \\ S \subseteq N \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{rational polyhedral} \\ \text{cones} \\ \sigma \subseteq N_{\mathbb{R}} \end{array} \right\}$$

• Fix a lattice N, $M = N^\vee = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

• **Def**: Given a rational polyhedral cone $\sigma \subseteq N$ its dual is

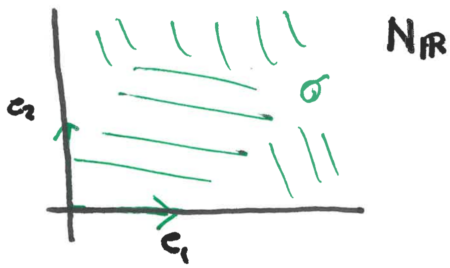
$$\sigma^\vee = \{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \quad \forall u \in \sigma \}$$

• **Lemma**: If $\sigma \subseteq N_{\mathbb{R}}$ is a RPC then

1) $\sigma^\vee$ is a rational polyhedral cone

2) $(\sigma^\vee)^\vee \cong \sigma$ under the indentifaction $(M^\vee)^\vee \cong N$.

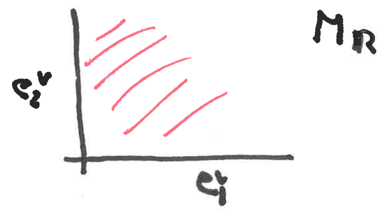
• **Ex**: $\sigma = \mathrm{cone}(e_1, e_2) \subseteq \mathbb{R}^2$



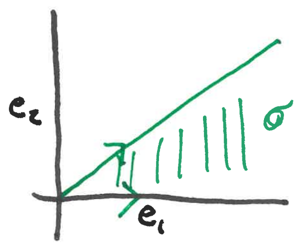
$\sigma^\vee = \{(m_1, m_2) \mid \langle m, u\rangle \geq 0 \ \forall u \in \sigma\}$

$= \{(m_1, m_2) \mid \langle m, e_1\rangle \geq 0 \ \langle m, e_2\rangle \geq 0\}$

$= \{(m_1, m_2) \mid m_1, m_2 \geq 0\}$

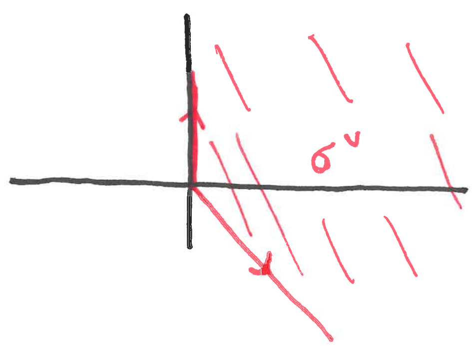


• **Ex**: $\sigma = \mathrm{cone}\left((1,0), (1,1)\right) \subseteq \mathbb{R}^2$

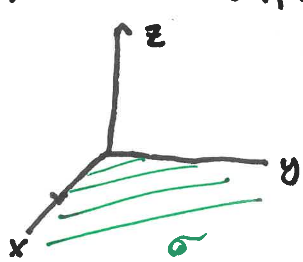


$\sigma^\vee = \left\{ m \in M_{\mathbb{R}} \;\middle|\; \begin{array}{l} \langle m, (1,0)\rangle \geq 0 \\ \langle m, (1,1)\rangle \geq 0 \end{array} \right\}$

$= \left\{ (m_1, m_2) \;\middle|\; \begin{array}{l} m_1 \geq 0 \\ m_1 + m_2 \geq 0 \end{array} \right\}$

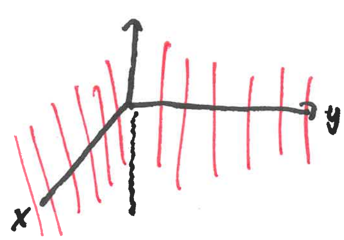


• **Ex**: $\sigma = \mathrm{cone}(e_1, e_2) \subseteq \mathbb{R}^3$



$\sigma^\vee = \{ m \mid \langle m, e_1\rangle \geq 0, \langle m, e_2\rangle \geq 0 \}$

$= \{ (m_1, m_2, m_3) \mid m_1, m_2 \geq 0 \}$



• Facts: Let $\sigma \subseteq N_{\mathbb{R}}$ be a RPC then

① If $\sigma = \text{cone}(u_1, \ldots, u_r)$ w/ $u_i \in N$ then

$$\sigma^\vee = \{ m \in M_{\mathbb{R}} \mid \langle m, u_i \rangle \geq 0 \}_{i=1,r}$$

~~②~~

• Def: A RPC $\sigma \subseteq N_{\mathbb{R}}$ is strongly convex

$\Leftrightarrow$ σ contains no positive dim. subspace

$\Leftrightarrow$ $\sigma \cap (-\sigma) = \{0\}$

$\Leftrightarrow$ $\dim \sigma^\vee = \dim N_{\mathbb{R}}$.

• Def: Given a rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$.

$$S_\sigma = \sigma^\vee \cap M$$

$$k[S_\sigma] = k[\sigma^\vee \cap M].$$

• Thm: Fix a lattice N. there is a bijection

$$\left\{ \begin{array}{c} \text{RPC } \sigma \subseteq N_{\mathbb{R}} \\ \text{strongly convex} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{normal toric varieties} \\ \text{w/} \\ \text{torus} = T_N \end{array} \right\}$$

key $\sigma \subseteq N$ strongly convex $\Leftrightarrow$ $\sigma^\vee \subseteq M$ full dim

$\Leftrightarrow$ $\mathbb{Z} S_\sigma \cong M$

• **Def**: Let V be an affine variety. The dimension of V is the supremum of lengths of all chains

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V$$

where V_i are distinct non-empty irred. closed subsets of V

• **Lemma**: $\dim V = \dim k[V]$

Krull dim = sup of length of prime ideals $P_0 \subsetneq P_1 \cdots \subsetneq P_d$

• **Lemma**: Let V be an irred. affine variety. If $U \subseteq V$ is an non-empty open subset then $\dim U = \dim V$.

Pf: Use U is dense and the def. of subspace top. ∎

• **Cor**: 1) Let S be an affine semigroup the

$$\dim X_S = \dim T_{\mathbb{Z}S} = \dim k[\mathbb{Z}S] = \operatorname{rk} \mathbb{Z}S.$$

2) If $\sigma \subseteq N_{\mathbb{R}}$ is a strongly convex RPC the

$$\dim X_\sigma = \dim \sigma.$$

• **Pf**: This reduces to showing $k[x_1^{\pm}, \dots x_n^{\pm}] = n$, which is non-trivial and assume ∎

• **Ex** $S = \mathbb{N}\{2,3\}$ Then X_S has dim $= 1$ as expected.

• **Def**: Let $m \in M_{\mathbb{R}}$ then

$$H_m = \{ u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0 \} \subseteq N_{\mathbb{R}} \qquad \leftarrow \text{hyperplane}$$

$$H_m^+ = \{ u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \} \subseteq N_{\mathbb{R}} \qquad \leftarrow \text{closed half space}$$

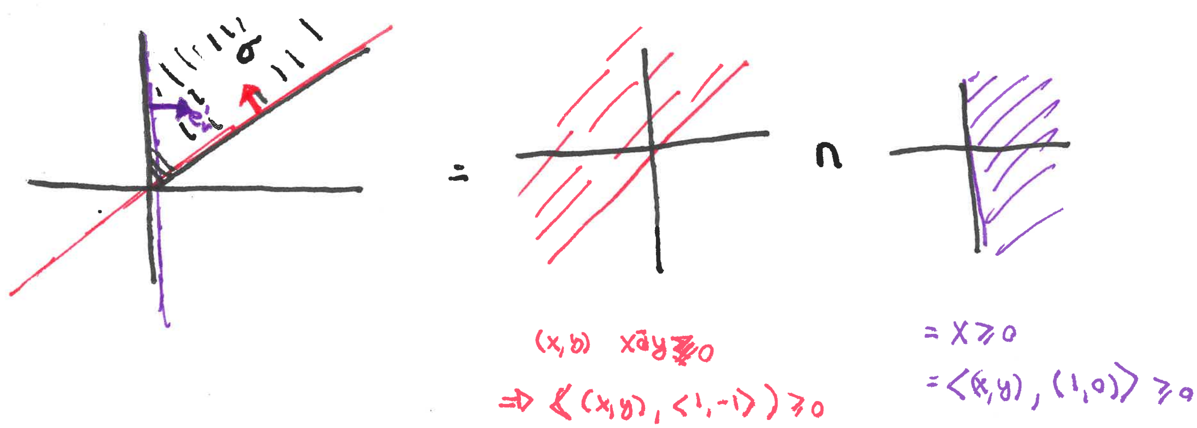
• **Prop**: Let $\sigma \subseteq N_{\mathbb{R}}$ be a RPC.

① $\sigma \subseteq H_m^+ \Leftrightarrow m \in \sigma^\vee$

② If $m_1, \dots m_s$ generate $\sigma^\vee$ then

$$\sigma = H_{m_1}^+ \cap \dots \cap H_{m_s}^+$$

③ If $\sigma = H_{m_1}^+ \cap \dots \cap H_{m_s}^+$ then $\sigma^\vee = \text{cone}(m_1, \dots m_s)$



• **Def**: A face of a cone σ is a subset $\tau \subseteq \sigma$ such that

1) τ is a cone

2) $x, y \in \sigma$ and $x + y \in \tau \Rightarrow x, y \in \tau$

Lemma: Let $\sigma \subseteq N_{\mathbb{R}}$ be a RPC. TFAE

① $\tau \subseteq \sigma$ is a face

② $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$

Lemma: Let $\sigma \subseteq N_{\mathbb{R}}$ be a RPC.

1) every face of σ is a RPC

2) The intersection of two faces is again a face

3) The face of a face ~~of σ~~ is then a face of σ.

Prop: Let $\sigma \subseteq N_{\mathbb{R}}$ be a s.c.RPC. If $\tau \subseteq \sigma$ is a face and ~~then~~ $\tau = \sigma \cap H_m$ then the inclusion

$$\sigma^\vee \cap M \xhookrightarrow{\ \iota\ } \tau^\vee \cap M$$

induces a map of rings which factors as

$$\begin{tikzcd}
k[S_\sigma] \arrow[r, hook, "\iota^\#"] \arrow[d, dashed] & k[S_\tau] \\
k[S_\sigma][\frac{1}{\chi^m}] \arrow[ur, "\sim", "\tilde{\iota}"'] &
\end{tikzcd}$$

Pf: By definition $\langle u,m\rangle = 0$ for all $u \in \tau$ so $\langle u,-m\rangle \geq 0$ $\forall\, u \in \tau$ so $m, -m \in \tau$ and so χ^m and χ^{-m} are in $k[S_\tau]$. Hence by the UPL $\iota^\#$ factors through the localization map. Injectivity of $\tilde{\iota}$ is for free.

$\underset{\sim}{\text{Pf}}$: (cont): Let $v \in S_\tau$ WTS χ^v is in the image of $\tilde{\imath}$, which is equivalent to saying $(\chi^m)^a \chi^v$ is in the image of $\imath$ for large enough a. But this is equivalent to

$$am + v \in \sigma^\vee \text{ for } a \gg 0. \iff \langle am+v, u\rangle \geq 0 \quad \forall u \in \sigma \quad a \gg 0$$

But since σ is RPC we only need

$$a\langle m, u_i\rangle + \langle v, u_i\rangle \geq 0 \text{ for}$$

a finite set of generators $u_1, \dots, u_t$ of σ. But if $u_i \in \tau$ the $\langle m, u_i\rangle = 0$ and by definition $m \in \sigma^\vee$ so $\langle m, u_i\rangle \geq 0$ so a large than $\max \langle v, u_i\rangle$ works $\blacksquare$

$$\tau^\vee \cap M = (\sigma^\vee \cap M) + \mathbb{Z}_{\geq 0}(m).$$

- Recall points of an affine toric variety = semigroup maps to $\mathbb{R}$

- $\underset{\sim}{\text{Def}}$: For a face $\tau \subseteq \sigma$ define the point $x_\tau \in U_\sigma$ to be

$$S_\sigma \xrightarrow{\ \gamma\ } \mathbb{R} \qquad\qquad \mathbb{R}[S_\sigma] \xrightarrow{\ x_\tau^*\ } \mathbb{R}$$

$$s \longmapsto \begin{cases} 1 & s \in \tau^\perp \cap S_\sigma \\ 0 & \text{else} \end{cases} \qquad\qquad \chi^s \longmapsto \begin{cases} 1 & s \in \tau^\perp \cap S_\sigma \\ 0 & \text{else} \end{cases}$$

$$\tau^\perp = \{ m \in M_{\mathbb{R}} \mid \langle m, u\rangle = 0 \ \forall u \in \tau \}$$

Lemma: This is a semigroup map

Pf: Need to show

$$\jmath(s+s') = \jmath(s)\cdot\jmath(s')$$

This is equivalent to $s, s' \in \tau^\perp \iff s+s' \in \tau^\perp$. For this

$$\langle s+s', u\rangle = \langle s, u\rangle + \langle s', u\rangle$$

Need $RH = 0 \iff LHS = 0$ which is true since as $s, s' \in \sigma^\vee$ we know $\langle s,u\rangle \geq 0$ and $\langle s',u\rangle \geq 0$ $\forall u \in \tau \subseteq \sigma$. $\blacksquare$

• Def: The torus orbit associated to τ is

$$O(\tau) = T_N \cdot x_\tau \subseteq U_\sigma$$

Recall: The action of T_N on $X_S = U_\sigma$ was

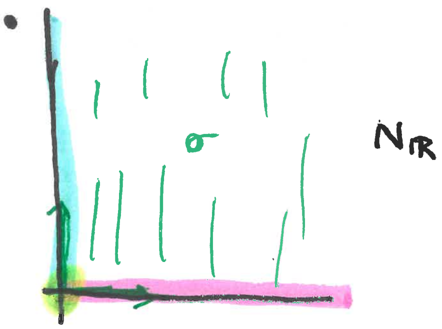
$$(M \xrightarrow{t} R^\times,\ S_\sigma \xrightarrow{x} R) \longmapsto \left(\begin{matrix} S_\sigma \longrightarrow R \\ s \longmapsto t(s)x(s)\end{matrix}\right)$$

Fix $t \in T_N$ given by $M \xrightarrow{t} R^\times$ then

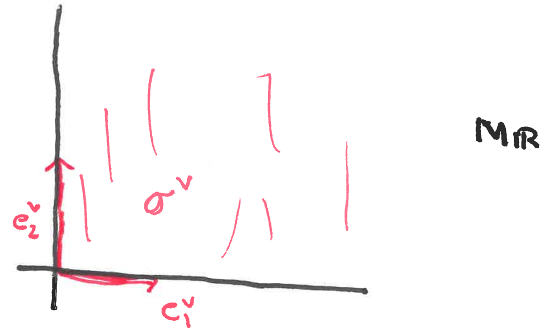
$$t\cdot x_\tau(s) = \begin{cases} t(s) & s \in \tau^\perp \cap S_\sigma \\ 0 & 0 \end{cases}$$

Since $t(s) \neq 0$ for all $s \in S_\sigma$

$$O(\tau) = \left\{ S_\sigma \xrightarrow{x} R \;\middle|\; \begin{matrix} x(s) \neq 0 \\ \iff s \in \tau^\perp \cap S_\sigma \end{matrix} \right\}$$



$\sigma = \text{cone}(e_1, e_2)$



$\sigma^\vee = \text{cone}(e_1^\vee, e_2^\vee)$

Faces

$\tau_1 = \text{coe}(e_1)$

$\tau_2 = \text{cone}(e_2)$

$0 = \text{cone}(0)$

$\boxed{\tau_1^\perp} = \{ m \in M_{\mathbb{R}} \mid \langle m, \lambda e_1 \rangle = 0 \}$

$= \text{Span}(e_2^\vee)$

$\boxed{\tau_1^\vee} = \{ m \in M_{\mathbb{R}} \mid \langle m, e_1 \rangle \geq 0 \} = \{ m \mid m_1 \geq 0 \}$

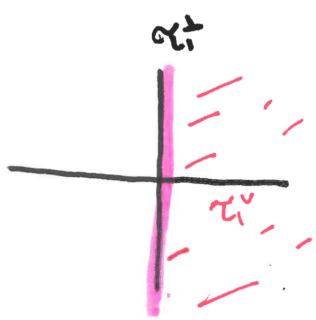
$\boxed{0^\perp} = \{ m \in M \mid \langle m, 0 \rangle = 0 \}$

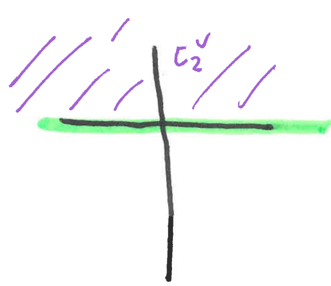
$= M_{\mathbb{R}}$

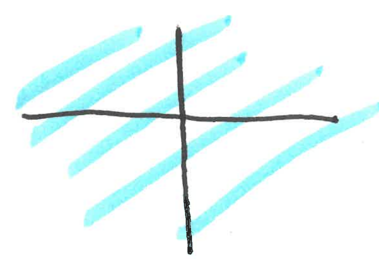
$\boxed{0^\vee} = M_{\mathbb{R}}$

$\boxed{\tau_2^\perp} = \{ m \in M_{\mathbb{R}} \mid \langle m, e_2 \rangle = 0 \} = \text{span}(e_1^\vee)$

$\boxed{\tau_2^\vee} = \{ m \in M \mid \langle m, e_2 \rangle \geq 0 \} = \{ m \mid m_2 \geq 0 \}$







Notice the choice of $m \in M$ s.t. $H_m \cap \sigma = \tau$ is exactly $\tau^\perp$

$\tau^\perp$ = the normals to the hyperplanes defining τ.

• Ex: $\sigma = \text{cone}(e_1, e_2)$

$\sigma^\vee = \text{cone}(e_1^\vee, e_2^\vee)$

$S_\sigma = \mathbb{N}^2 \subseteq \mathbb{Z}^2$

$k[S_\sigma] = k[x,y]$

$U_\sigma = \mathbb{A}^2$

A point $p = (x,y) \in \mathbb{A}^2$ corresponds to

$$S_\sigma \longrightarrow K$$
$$(s_1, s_2) \longmapsto x^{s_1} y^{s_2}$$

The torus T_N is $(k^\times)^2 \subseteq \mathbb{A}^2$ corresponds to the same morphism.

$$t \cdot p = (s_1 s_2) \longmapsto t_1^{s_1} t_2^{s_2} x_1^{s_1} x_2^{s_2} = (t_1 x_1)^{s_1} (t_2 x_2)^{s_2}$$

$\tau_1 \rightsquigarrow x_{\tau_1}$ = want (x,y) s.t. the map

$$S_\sigma \longrightarrow k \qquad (s,t) \longmapsto x^s y^t = \begin{cases} 1 & \text{iff } \underset{\substack{\| \\ (0,\lambda)}}{(s,t)} \in \text{span}(e_2^\vee) \\ 0 & \text{else} \end{cases}$$

$= (s,t) \longmapsto 0^s 1^t$ with 0^0 convention

$\tau_2 \rightsquigarrow x_2$. $S_\sigma \longrightarrow k \qquad (s,t) \longmapsto x^s y^t = \begin{cases} 1 & \text{iff } (s,t) = (\lambda, 0) \\ 0 & \text{else} \end{cases}$

$$O(\tau_1) = \{(0,y) \mid y \in k^\times\} \qquad \{(x,0) \mid x \in k^\times\} = O(\tau_2)$$

$$O(0) = \{(x,y) \in k^\times\}$$

• Lemma: Let $\tau \subseteq \sigma$ be a face

$$O(\tau) = \left\{ S_\sigma \xrightarrow{\gamma} k \;\middle|\; \begin{array}{l} \gamma(m) \neq 0 \\ \Leftrightarrow m \in \tau^\perp \cap M \end{array} \right\} \subseteq M$$

$$\simeq \text{Hom}_{\mathbb{Z}}(\tau^\perp \cap M, k^\times)$$

this should be the points of some torus!

• **PF** : (cont): Now $N_\tau \hookrightarrow N$ so $\underset{\cong\atop M}{\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})} \longrightarrow \underset{\|\atop M_\tau}{\mathrm{Hom}_{\mathbb{Z}}(N_\tau,\mathbb{Z})}$

the kernel is precisely

$$\{ m \in M \mid \langle m,u \rangle = 0 \ \forall u \in N_\tau \} = \tau^\perp \cap M!$$

Hence $T_{N_\tau} = \mathrm{Stab}_{T_N}(x_\tau)$.

∎

• **COR** : $O(\tau) \cong T_N / T_{N_\tau} \cong T_{N(\tau)}$

• **PF** : Orbit-Stabilizer!

∎

• **COR** : $\dim O(\tau) = \dim T_N - \dim T_{N_\tau} = \dim N_{\mathbb{R}} - \dim \tau$ ∎

• **Thm** : (Orbit Cone Correspondence): Let $\sigma \subseteq N_{\mathbb{R}}$ be a SCRPC.

① There is an inclusion reversing bijection

$$\left\{ \begin{array}{c} \text{faces of} \\ \sigma \text{ codim } k \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Orbits of} \\ T_N \text{ on } X \text{ of} \\ \dim = k \end{array} \right\}$$

② If $\tau \subseteq \sigma$ is a face then

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \gamma} O(\gamma)$$

③ If $\tau, \gamma \subseteq \sigma$ are faces then

$$O(\gamma) \subseteq \overline{O(\tau)} \iff \tau \preceq \gamma.$$

• **Def**: Given a cone $\tau \subseteq N$ let N_τ be the sublattice spanned by $\tau \cap N$. Set $N(\tau) = N/N_\tau$.

• **Ex**: $\mathbb{A}^2$

$$\tau_1 = \operatorname{cone}(e_1) \Rightarrow N_{\tau_1} = \mathbb{Z}(1,0) \rightsquigarrow N(\tau_1) \cong \mathbb{Z}\langle e_2 \rangle$$

$$\tau_2 = \operatorname{cone}(e_2) \Rightarrow N_{\tau_2} = \mathbb{Z}(0,1) \rightsquigarrow N(\tau_2) \cong \mathbb{Z}\langle e_1 \rangle$$

$$\tau_0 = \operatorname{cone}(0,0) \Rightarrow N_0 = \mathbb{Z}(0,0) \rightsquigarrow N(0) = \mathbb{Z}\langle e_1, e_2 \rangle$$

• **Lemma**: Fix a cone $\tau \leq \sigma$, the action of T_N on U_σ induces an algebraic map

$$\begin{aligned} T_N &\xrightarrow{\mu_\tau} U_\sigma \\ t &\longmapsto t \cdot x_\tau \end{aligned}$$

this map has image $O(\tau)$ and

$$\operatorname{Stab}_{T_N}(x_\tau) = \mu_\tau^{-1}(x_\tau) = \{ t \in T_N \mid t\, x_\tau = x_\tau \} \cong T_{M_\tau}$$

• **PF**: Surjectivity is clear from the definition of $O(\tau)$.

To compute the stabilizer

$$(t \cdot x_\tau)(m) = \begin{cases} t(m) & m \in \tau^\perp \cap M \\ 0 & \text{else} \end{cases}$$

Thus $(t \cdot x_\tau)(m) = x_\tau(m) \Leftrightarrow t(m) = 1$ for all $m \in \tau^\perp \cap M$

$$\operatorname{Stab}_{T_N}(x_\tau) = \{ t \in T_N \mid t(m) = 1 \ \forall m \in \tau^\perp \cap M \}$$

- **Lemma**: Let $Z \subseteq U_\sigma$ be closed subvariety fixed under the action of T_N, then $I_Z \subseteq k[S_\sigma]$ is homogeneous w/r/t the M-grading.

■

- **Lemma**: If $Z \subseteq U_\sigma$ is a closed subvariety fixed under the action of T_N then $I_Z \subseteq k[S_\sigma]$ is a monomial ideal.

- **Pf**: The graded pieces of $k[S_\sigma]$ are 1-dim spanned by monomials ■

- **Lemma**: Let $P \subseteq k[S_\sigma]$ be a prime homogeneous ideal. Define $F(P) = \{ m \in S_\sigma \mid \chi^m \notin P \}$. If $m, m' \in S_\sigma$ such that $m + m' \in F(P)$ then $m, m' \in F(P)$ & $F(P)$ is a semigroup.

- **Pf**: Suppose $m + m' \in F(P) \Leftrightarrow \chi^{m+m'} = \chi^m \chi^{m'} \notin P$. ~~[illegible]~~

If $m \notin F(P)$ then $\chi^m \in P \Rightarrow \chi^{m'} \cdot \chi^m = \chi^{m'+m} \in P$, which is a contradiction. Thus $m \in F(P)$ and the same argument $\Rightarrow m' \in F(P)$.

Further $1 = \chi^0 \notin P \Rightarrow 1 \in F(P)$ and if $m, m' \in F(P)$ then $\chi^m \notin P$ and $\chi^{m'} \notin P$

Prime $\Rightarrow \chi^m \chi^{m'} \notin P$.

■

- **Ex**: $\mathbb{A}^2$

$\overline{O(\tau_1)} = \{(0,y)\} = V(x)$ $\quad x = $ "e_1"

$\overline{O(\tau_2)} = \{(x,0)\} = V(y)$ $\quad y = $ "e_2"

$\overline{O(0)} = \mathbb{A}^2 = V(x,y)$. $\quad x, y = $ "e_1, e_2"

- **Pf of Thm:**

We constructed $\tau \longmapsto O(\tau)$. For the converse let $Z \subseteq U_\sigma$ be the closure of an T_N-orbit. Since orbits are T_N stable and irreducible so is Z. Hence $I_Z \subseteq k[S_\sigma]$ is a prime monomial ideal and $F(I_Z) \subseteq S_\sigma$ is a face of S_σ. There is a bijection between faces of S_σ and faces of σ, hence there exists a unique $\tau \leq \sigma$ such that $\tau^\perp \cap M = F(I_Z)$. One checks these are inverses.

$$
\begin{aligned}
\text{If } O(\gamma) \subseteq \overline{O(\tau)} \iff & \; \mathbb{I}(\overline{O(\tau)}) \subseteq \mathbb{I}(\overline{O(\gamma)}) \\
\iff & \; \gamma^\perp \cap M \subseteq \tau^\perp \cap M \\
\iff & \; \tau \leq \gamma .
\end{aligned}
$$

$\blacksquare$

- **COR**: There is a bijection

$$
\left\{ \begin{matrix} \text{monomial prime} \\ \text{ideals} \\ k[S_\sigma] \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{faces} \\ \tau \leq \sigma \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} T_N\text{-orbit} \\ \text{closures} \end{matrix} \right\}.
$$

- **COR**: There is a fixed point of the T_N action on $U_\sigma \iff \dim \sigma = \dim N$. In which case the fixed point is unique and defined by.

$$
\begin{aligned}
S_\sigma &\longrightarrow \mathbb{R} \\
m &\longmapsto \begin{cases} 1 & m = 0 \\ 0 & \text{else.} \end{cases}
\end{aligned}
$$

- Let V be an irreducible affine variety.

- Def : The local ring of V at a point $p \in V$ is

$$k[V]_p = \left\{ \frac{f}{g} \in k(V) \;\middle|\; \begin{matrix} f, g \in k[V] \\ g(p) \neq 0 \end{matrix} \right\}$$

the maximal ideal at p is

$$m_p = \{ \varphi \in k[V]_p \mid \varphi(p) = 0 \}.$$

- Notice if $\varphi \in k[V]_p$ then φ is a function on some open neighborhood of p.

- Def: An affine $\overset{\text{irred.}}{\text{variety}}$ V is smooth at p iff

$$\dim V = \dim_k \left(m_p / m_p^2 \right)$$

- Thm: Let $V \subseteq \mathbb{A}^n$ be an affine variety w/ $V = \mathbb{V}(F_1, \ldots, F_r)$ - assume the F_i generator $\mathbb{I}_V$ - define the tangent space of V at p

$$T_pV = \mathbb{V}\left(dF_1|_p^{(x-p)}, \ldots, dF_r|_p^{(x-p)} \right)$$

where $dF_i(x-p) = \sum \frac{\partial F_i}{\partial F_j}(x_j - p_j)$. Then

① T_pV is independent of the choice of F_i's

② $\dim T_pV = \dim_k (m_p / m_p^2)$

③ The set of singular points of V is

$$\operatorname{Sing}(V) = V \cap \mathbb{V}\left(\begin{matrix} \text{codim} \\ \times \text{codim} \end{matrix} \text{ minor } \left| \frac{\partial F_i}{\partial x_j} \right| \right)$$

• **Ex**: $\mathbb{A}^n$ $\quad F_i = 0, \quad dF_i = 0 \quad \Rightarrow T_pV = \mathbb{A}^n$

$\Rightarrow \mathbb{A}^n$ is smooth.

$P = (0, \dots, 0)$

$m_p = \langle x_1, \dots, x_n \rangle \qquad m_p^2 = \langle x_1^2, x_1x_2, \dots, x_n^2 \rangle \qquad m_p/m_p^2 = \{\bar{x}_1, \dots, \bar{x}_n\}$

• **Ex**: $\mathbb{G}_m \subseteq \mathbb{A}^2 \qquad \mathbb{V}(xy-1)$

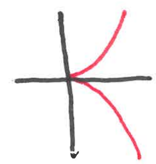
$\mathrm{Sing}(\mathbb{G}_m) = \mathbb{V}\left(xy-1, \begin{vmatrix} y \\ x \end{vmatrix}_{1\times 1}\right) = \mathbb{V}(xy-1, x, y) = \emptyset$

More generally $\mathbb{G}_m^n \subseteq \mathbb{A}^{n+1}$.

• **Ex**: $\mathbb{A}^r \times (\mathbb{G}_m^{n-r})$ is smooth.

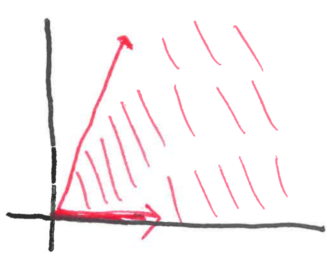
• **Ex**: $\mathbb{V}(x^3 - y^2) \subseteq \mathbb{A}^2$

$\mathrm{Sing}(V) = \mathbb{V}(x^3 - y^2, 3x^2, 2y) = \{(0,0)\}$



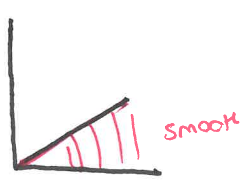
• **Def**: A strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ iff th. can be generated by elements that are a $\mathbb{Z}$-basis for N iff det of the rays is ± 1.

• **Ex**: $\mathrm{Cone}((1,0),(1,d))$



smooth $\Leftrightarrow$

$$\begin{vmatrix} 1 & 1 \\ 0 & d \end{vmatrix} = d = \pm 1$$



• **Thm** U_σ is smooth $\Leftrightarrow$ σ is smooth

Pf $\Leftarrow$ $\sigma \cong \mathbb{R}e_1 \oplus \cdots \mathbb{R}e_t \Rightarrow \sigma^\vee \cap M = \mathbb{N}e_1^\vee \oplus \cdots \mathbb{N}e_t^\vee \oplus \mathbb{Z}e_{t+1}^\vee \oplus \cdots \mathbb{Z}e_n^\vee$

and so $k[S_\sigma] \cong k[x_1, \dots x_t, x_{t+1}^{\pm}, \dots x_n^{\pm}]$

giving $U_\sigma \cong \mathbb{G}_m^{n-t} \times \mathbb{A}^t$.

$\Rightarrow$ Assume $\sigma \subseteq N$ has full dimension. Hence σ has a unique T_N fixed point x_σ and $\mathbb{I}(x_\sigma)$ is the prime monomial ideal

$$\mathbb{I}(x_\sigma) = \langle \chi^m \mid m \in S_\sigma \setminus 0 \rangle$$

Since U_σ is smooth it is smooth at x_σ and so

$$\dim \mathfrak{m}_{x_\sigma}/\mathfrak{m}_{x_\sigma}^2 = \dim U_\sigma = n.$$

But in an affine variety $\mathfrak{m}_{x_\sigma} = \mathbb{I}(x_\sigma)$. A basis for $\mathfrak{m}_{x_\sigma}$ is $\{\chi^m \mid m \in S \setminus 0\}$ and a basis for $\mathfrak{m}_{x_\sigma}^2$ is $\{\chi^m \mid \begin{smallmatrix} m \in S \setminus 0 \\ m = a + b \end{smallmatrix}\}$ hence

$$\mathfrak{m}/\mathfrak{m}_{x_\sigma}^2 = \operatorname{span}\{\chi^{s_1}, \dots \chi^{s_n}\}$$

where $s_1, \dots s_n \in S_\sigma \setminus 0$ are indecomposible elements. Hence $S = \mathbb{N}\{s_1, \dots s_N\}$ as $\sigma^\vee$ is full dim $\Rightarrow S$ generates M as a $\mathbb{Z}$-module and so $s_1, \dots s_N$ are a $\mathbb{Z}$-basis for M. Now dualize.

If U_σ is not full dim reduce to $U_\sigma = U_{\sigma, N_\sigma} \times T$.

• **Def:** Let $S \subseteq \mathbb{Z}S$ be an affine semigroup. The Hilbert Basis of S is the set of indecomposable elements of S

- **COR**: If S is an affine semigroup X_S is an affine toric variety.

- By our work on lattice ideals we also know every such variety has an embedding into $\mathbb{A}^n$

$$X_S \xhookrightarrow{\pi_A} \mathbb{A}^t$$
$$p \longmapsto (\chi^{m_1}(p), \dots \chi^{m_t}(p))$$

where $\chi^{m_i} \in k[S]$ and $m_1, \dots m_t$ are generators for S. The image of this is given by $\mathbb{V}(I_L)$ where I_L is the lattice ideal

$$I_L = \left\langle y^u - y^v \;\middle|\; \begin{matrix} u, v \in \mathbb{N}^t \\ u - v \in L \end{matrix} \right\rangle$$

For $L = \ker(\mathbb{Z}^t \xrightarrow{A} M)$ with A induced by $m_1, \dots m_t$.

- Two-Fixes From Last Week

well really just proofs I skipped.

- **Lemma**: Let $L \subseteq \mathbb{Z}^t$ be a sub lattice.

$$\underbrace{\left\langle y^{\ell^+} - y^{\ell^-} \;\middle|\; \ell \in L \right\rangle}_{I} = \underbrace{\left\langle y^u - y^v \;\middle|\; \begin{matrix} u, v \in \mathbb{N}^t \\ u - v \in L \end{matrix} \right\rangle}_{J}$$

for $\ell^+ = \sum_{\ell_i > 0} \ell_i e_i$ and $\ell^- = \sum_{\ell_i < 0} -\ell_i e_i$.

• *PF*: We show both inclusions, on generators

$\boxed{I \subseteq J}$: Take $y^{\ell^+} - y^{\ell^-}$ for $\ell \in L$. Then $\ell^+, \ell^- \in \mathbb{N}^t$ by construction and $\ell = \ell^+ - \ell^- \in L$ so $y^{\ell^+} - y^{\ell^-} \in J$.

$\boxed{I \supseteq J}$: Take $y^u - y^v \in J$ for $u, v \in \mathbb{N}^t$ and $u - v \in L$.

Define $\ell = u - v \in L$. Cordinatewise we have that

$$(*) \qquad u \geq \ell^+ \qquad \text{and} \qquad v \geq \ell^-$$

because ℓ^+ picks out the cordinates where $u_i \geq v_i$ and then in this case $\ell_i^+ = u_i - v_i \leq u_i$. Similarly ℓ^- picks out cordinates where $v_i \geq u_i$ and then $\ell_i^- = -(u_i - v_i) = v_i - u_i \leq v_i$.

Note $\ell^+ \neq u$ and $\ell^- \neq v$ as we might have cancelation in $u-v$
$u = (3,1)$ $v = (1,3)$ $\ell = (2,-2)$
$\ell^+ = (2,0)$, $\ell^- = (0,2)$
this equality is only true if u and v have disjoint support

Define $w = u - \ell^+$ not by $(*)$ $w \in \mathbb{N}^t$ further we also have

$$w = v - \ell^-$$

But then we can write our elements

$$y^u - y^v = y^{w+\ell^+} - y^{w+\ell^-} = y^w\left(y^{\ell^+} - y^{\ell^-}\right) \in I$$

Note y^w is the cancelation terms ∎

$$y_1^3 y_2^1 - y_1^1 y_2^3 = y_1 y_2 \left(y_1^2 - y_2^2\right)$$

$w = (1,1)$ $\ell^+ = (2,0)$ $(0,2) = \ell^-$.

• **Ex**: Consider the lattice generated by the row span

$$L = \mathrm{rowspan}_{\mathbb{Z}} \begin{pmatrix} 1 & -2 & 1 \\ 4 & -3 & 0 \end{pmatrix} \subseteq \mathbb{Z}^3$$

One checks the cokernel of the lattice L is torsion free since SNF

$$\begin{pmatrix} 1 & -2 & 1 \\ 4 & -3 & 0 \end{pmatrix} \xrightarrow{\text{SNF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So L is saturated sublattice of $\mathbb{Z}^3$. Consider the ideal generated by

$$J = \langle y_1 y_3 - y_2^2 \, , \, y_1^4 - y_2^3 \rangle \subseteq \mathbb{R}[y_1, y_2, y_3]$$

this corresponds to only taking the binomials for our 2 lattice generators.

Let us compare J ~~to~~ I_L. Notice

$$\ell = (4,-3,0) - 2(1,-2,1) = (2,1,-2) \in L.$$

Thus, $y^{\ell^+} - y^{\ell^-} = y_1^2 y_2 - y_3^2 \in I_L$. I claim $y_1^2 y_2 - y_3^2 \notin J$.

Notice $J \subseteq \langle y_1, y_2 \rangle$ but cleary $y_1^2 y_2 - y_3^2 \in \langle y_1, y_2 \rangle$.

Thus lattice generators for L do <u>not</u> give generators for I_L even when L is saturated ($\Leftrightarrow I_L$ prime).

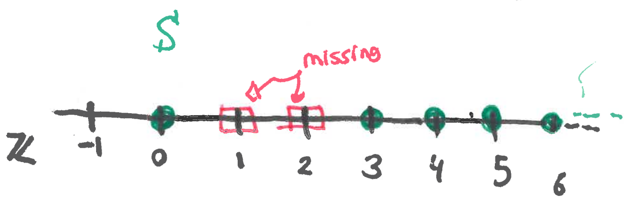
• Notice geometrically everything in J vanishes at $(0,0,1)$ but our given ~~monomial~~ binomial does <u>not</u>!

- Let's look at $X_{S'}$ dirrectly!

- **Ex:** Notice that we have a SES

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & -2 & 1 \\ 4 & -3 & 0 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}} \mathbb{Z} \longrightarrow 0$$

Hence here $S' = \mathbb{N}\{3, 4, 5\} \subseteq \mathbb{Z}$, note this is not a saturated semigroup ~~we took a non minimal generating set~~ for S' but that is still a map, and remember surjections of lattices do not see saturation of S'.



Our embedding of $X_{S'}$ is then

$$\mathbb{C}[y_1, y_2, y_3] \xrightarrow{\pi_A^*} \mathbb{C}[t^3, t^4, t^5] \cong \mathbb{C}[\chi^3, \chi^4, \chi^5]$$

$$y_1 \longmapsto t^3$$
$$y_2 \longmapsto t^4$$
$$y_3 \longmapsto t^5$$

which geometrically is the monomial curve

$$X_{S'} \overset{\pi_A}{\hookrightarrow} \mathbb{A}^3$$
$$t \longmapsto (t^3, t^4, t^5)$$

But <u>No!</u> t is <u>not</u> a regular function on $X_{S'}$. The map is $p \longmapsto (\chi^3(p), \chi^4(p), \chi^5(p))$.

<u>But</u> there is an open dense subset of $X_{S'}$ where t is a cordinate! $T_N = \mathbb{C}^*$. Coming from $\mathbb{C}[S'] \hookrightarrow \mathbb{C}[M] = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[t, t^{-1}]$. So what we actually have

$$\begin{tikzcd}
\mathbb{C}[S] \arrow[r, hook, "\pi_A^*"] & \mathbb{C}[M] \\
\mathbb{C}[y_1, y_2, y_3] \arrow[u, "i^*"] \arrow[ur, dashed, "\overline{\Phi}_A^*"'] \arrow[u, phantom, shift left=6, "?"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{S'} \arrow[r, hook, "\pi_A"] & \mathbb{A}^3 \\
T_{\mathbb{Z}} \arrow[u, hook, "i"] \arrow[ur, dashed] \arrow[ur, "\overline{\Phi}_A"']
\end{tikzcd}$$

$$t \longmapsto (t^3, t^4, t^5)$$

• Ex : Notice $\mathrm{img}(\Phi_A) \neq \mathrm{img}(\pi_A)$ instead $\mathrm{img}(\Phi_A) \subseteq \mathrm{img}(\pi_A)$ is dense since $T_N \hookrightarrow X_{\mathcal{S}}$ is a dense open immersion.

• Thm : If $B \subseteq L$ is generators for L then

$$I_L = J_B : (x_1 \cdots x_n)^{\infty}$$