- Recall a affine T-variety is a pair (X, Δ) where X is an affine variety and Δ is an algebraic action of T on X:

$$T \times X \xrightarrow{\Delta} X$$

An affine T-embedding of an affine T-variety (X, Δ) is an ~~o~~ open immersion $T \overset{i}{\hookrightarrow} X$ that is T-equivariant with dense image in X.

$$\begin{tikzcd} T \times X \arrow[r, "\Delta"] & X \\ T \times T \arrow[u, hook, "id \times i"] \arrow[r, "\mu"] \arrow[ur, phantom, "2"] & T \arrow[u, hook, "i"'] \end{tikzcd}$$

- Def: A ~~toric~~ affine toric variety is a affine T-variety w/ a fixed affine T-embedding.

- Last time we proved:

- Prop: Let S be an affine semigroup $S \overset{j}{\hookrightarrow} \mathbb{Z}^n = M$ be the natural inclusion. Set $N = M^\vee$. There exists a unique T_N-variety structure on X_S, given by

$$\mathbb{k}[S] \xrightarrow{\Delta} \mathbb{k}[M] \otimes \mathbb{k}[S]$$
$$\chi^s \longmapsto \chi^{j(s)} \otimes \chi^s \quad = \text{"}\chi^s \otimes \chi^s\text{"}$$

such that the embedding induced by j,

$$T_N \overset{i}{\hookrightarrow} X_S$$

is a T_N-embedding.

- COR: If S is an affine semigroup then X_S is an affine toric variety.

- We would like a converse.

- Prop: Let M be a lattice and $N = M^\vee$. Let (X, Δ') be an affine T_N-variety and $\tau : T_N \hookrightarrow X$ an T_N-embedding. There exists a unique affine semigroup inclusion $S' \xhookrightarrow{i} M$ such that the induced map $\tau^\#$ factors as

$$\begin{tikzcd}
R[X] \arrow[r, "\tau^\#"] \arrow[dr, dashed, "\tilde{\tau}^\#"'] & R[M] \\
 & R[S'] \arrow[u, hook, "i^\#"']
\end{tikzcd}$$

where $\tilde{\tau}^\#$ is an isomorphism. Furthermore, this induce isomorphisms s.t. this commute

$$\begin{tikzcd}
R[X] \arrow[r, "\Delta'^\#"] \arrow[dr, phantom, "\circlearrowleft"] & R[M] \otimes k[X] \\
R[S'] \arrow[u, "\wr\ \tilde{\tau}^\#"] \arrow[r] & R[M] \otimes R[S'] \arrow[u, "\wr\ \mathrm{Id} \otimes \tilde{\tau}^\#"']
\end{tikzcd}$$

- Def: A morphism $\varphi : X \longrightarrow Y$ of affine varieties is dominant if $\mathrm{img}(\varphi)$ is dense in Y (in the Zariski topology).

equivalently $\overline{\mathrm{img}(\varphi)} = Y$.

- **Lemma:** Let X and why be affine varieties and $\varphi: X \longrightarrow Y$ a morphism. Then φ is dominant $\Longleftrightarrow$ $\varphi^{\#}: k[Y] \longrightarrow k[X]$ is injective.

Pf: $\Rightarrow$ suppose φ is dominant, i.e. $\operatorname{img}(\varphi) \subseteq Y$ is dense

Let $f \in \ker(\varphi^{\#})$ this means $Y \xrightarrow{f} k$ is a regular function such that $X \xrightarrow{\varphi} Y \xrightarrow{f} k$ is the zero function. This means $\forall y \in \operatorname{img}(\varphi)$ $f(y) = 0$. However, f is continuous in the Zariski topology and $\operatorname{img}(\varphi)$ is dense $\Rightarrow f = 0$ on all Y $\Rightarrow \ker(\varphi^{\#}) = 0 \Rightarrow$ injective.

$\Leftarrow$ suppose $\varphi^{\#}: k[Y] \longrightarrow k[X]$ is injective. It is enough to show every basic open $D(g) \subseteq Y$ has $\operatorname{img}(\varphi) \cap D(g) \neq \emptyset$. Recall $D(g) = \{p \in Y \mid g(p) \neq 0\}$ and these are a basis for the Zariski topology.

Since $\varphi^{\#}$ is injective $g \circ \varphi \not\equiv 0 \Longleftrightarrow \exists x \in X$ such that $g(\varphi(x)) \neq 0$

But this means $\exists y = \varphi(x)$ such that $y \in \operatorname{img}(\varphi)$ and $g(y) \neq 0 \Rightarrow$ $y \in D(g) \cap \operatorname{img}(\varphi)$.

$\blacksquare$

- As a recap

$$f \in \ker(\varphi^{\#}) \Longleftrightarrow \varphi^{\#}(f) = 0 \Longleftrightarrow f \circ \varphi \equiv 0 \Longleftrightarrow f(\varphi(x)) = 0 \text{ for all } x \in X$$

$$\Longleftrightarrow f(y) = 0 \quad \forall y \in \operatorname{img}(\varphi)$$

$$\overset{(*)}{\Longleftrightarrow} f(y) = 0 \quad \forall y \in \overline{\operatorname{img}(\varphi)}$$

$(*)$ is really the key and it uses that if $U \subseteq X$ is a subset then

$$\mathbb{I}(U) = \mathbb{I}(\overline{U})$$

which is equivalent to

$$\overline{U} = \mathbb{V}(\mathbb{I}(U)).$$

• Pf: Let (X, Δ') be an affine T_N-variety and $\tau: T_N \longrightarrow X$ a T_N-embedding. By the previous lemma since $img(\tau) \subseteq X$ is dense the induced map

$$k[X] \xhookrightarrow{\tau^{\#}} k[M]$$

is an injection. Define a subset $S \subseteq M$ as follows.

$$S = \{ m \in M \mid \chi^m \in img(\tau^{\#}) \} \xhookrightarrow{\kappa} M$$

since $\tau^{\#}$ is an injective ring map its image is a ring hence

① $1 = \chi^0 \in img(\tau^{\#}) \implies 0 \in S$

② $\chi^m, \chi^{m'} \in img(\tau^{\#})$
then $\chi^m \chi^{m'} = \chi^{m+m'} \in img(\tau^{\#}) \implies m, m' \in S$ then $m+m' \in S$

Therefore, S is a sub-semigroup of M. Note $k[S] \subseteq img(\tau^{\#})$

To get the other inclusion we $\boxed{\text{must}}$ use that τ is T_N-equivariant. * #1

Note the action of T_N on X induces a M-grading on $k[X]$ by

$$k[X]_m = \{ f \in k[X] \mid \Delta'^{*}(f) = \chi^m \otimes f \}$$

Recall the M-grading on $k[M]$ can similarly be defined by

$$k[M]_m = \{ f \in k[M] \mid \mu(f) = \chi^m \otimes f \}$$

Using this we will show the T_N-equivariance of $\tau \implies \tau^{\#}$ is a graded map w/r/t these gradings.

• <u>Pf</u>: (cont.) By definition τ being equivariant means diagrams

$$\begin{tikzcd}
T_N \times X \arrow[r, "\Delta'"] & X \\
T_N \times T_N \arrow[u, "id \times \tau"] \arrow[r, "\mu"] & T_N \arrow[u, "\tau"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k[X] \arrow[r, "\Delta'^{\#}"] \arrow[d, "\tau^{\#}"] & k[M] \otimes k[X] \arrow[d, "id \otimes \tau^{\#}"] \\
k[M] \arrow[r, "\mu^{\#}"] & k[M] \otimes k[M]
\end{tikzcd}$$

Let $f \in k[X]$ homogeneous of degree $m \in M \iff \Delta'^{\#}(f) = \chi^m \otimes f$

Chasing the diagram on the RHS gives.

$$\mu^{\#}(\tau^{\#}(f)) = (id \otimes \tau^{\#})(\Delta'^{\#}(f)) = (id \otimes \tau^{\#})(\chi^m \otimes f)$$

$$= \chi^m \otimes \tau^{\#}(f)$$

Hence $\deg(\tau^{\#}(f)) = m$ and so

$\tau^{\#}: k[X] \longrightarrow k[M]$ is a M-graded map. We now prove the other inclusion $img(\tau^{\#}) \subseteq k[S]$.

Let $g \in img(\tau^{\#})$

Pick $f \in k[X]$ such that $\tau^{\#}(f) = g$ and write $f = f_{m_1} + \dots + f_{m_t}$ for $f_{m_i} \in k[X]_{m_i}$. Then

$$g = \tau^{\#}(f) = \tau^{\#}(f_{m_1} + \dots + f_{m_t}) = \tau^{\#}(f_{m_1}) + \dots + \tau^{\#}(f_{m_t})$$

and the RHS are homogeneous elements of $k[M]$. $\Rightarrow \tau^{\#}(f_{m_i}) = c\chi^{m_i}$ for some $c \in k$, wlog $c \neq 0$. Thus, $\chi^{m_i} \in img(\tau^{\#}) \Rightarrow m_i \in S$ and $\chi^{m_i} \in k[S]$. Thus $g \in k[S]$ proving $k[S] = img(\tau^{\#})$.

• **PF** : (cont): Since $\tau^\#$ is injective we have thus shown we have

$$\begin{tikzcd}
R[X] \arrow[r, hook, "\tau^\#"] \arrow[dr, "\tilde{\tau}^\#", "\sim"'] & R[M] \\
 & R[S] \arrow[u, "\varphi_K"']
\end{tikzcd}$$

where $\tilde{\tau}^\#$ is just the restriction of $\tau^\#$ onto its image which is an isomorphism and φ_K is the inclusion induced by $K: S \hookrightarrow M$. In fact these are M-graded morphisms of k-algebras.

It remains to check ① S is affine, ② the actions are compatible, ③ $M \cong \mathbb{Z}S$. Since $R[X]$ is the coordinate ring of a variety, it is a f.g. k-algebra. Hence $R[S]$ is a f.g. k-algebra and $S \subseteq M$. The affine-ness of S is then the following lemma $\boxed{\#2}$

• **Lemma**: Let S be a semi-group. If $k[S]$ is f.g. as a k-algebra then S is finitely generated.

Pf: Pick k-algebra generators $f_1, \dots f_\ell \in k[S]$ so that $k[f_1, \dots, f_\ell] \cong k[S]$. For each f_i write $f_i = \sum_j c_j \chi^{m_{ij}}$ with $c_j \in k^\times$ and $m_{ij} \in S$. We claim $A = \{m_{ij}\}$ generates S. Pick $s \in S$. Since $\chi^s \in k[S] = k[f_1, \dots f_\ell] \Rightarrow$ $\chi^s = P(f_1, \dots f_\ell)$ but each monomial of $P(f_1, \dots f_\ell)$ is $\chi^{m_1}\chi^{m_2}\cdots\chi^{m_\ell}$ for $m_j \in A$ $\Rightarrow s \in \mathbb{N}(A)$. ∎

• **COR**: Let S be a semigroup

S is finitely generated $\Leftrightarrow$ $k[S]$ is a f.g. k-algebra

• **Pf**: (cont.): Therefore S' is finitely generated and thus affine since by construction S' is contained in a lattice M.

Let us now show there is a natural isomorphism. 3 $\mathbb{Z}S' \cong M$. Let $S' \xhookrightarrow{k} M$ be the inclusion given by construction. Let $j: S' \hookrightarrow \mathbb{Z}S'$ be the inclusion given by the Grothendieck group construction. By the universal property this gives a unique $\tilde{k}: \mathbb{Z}S' \longrightarrow M$.

$$\begin{tikzcd} S' \arrow[r, hook, "k"] \arrow[d, hook, "j"'] & M \\ \mathbb{Z}S' \arrow[ur, dashed, "\exists! \tilde{k}"'] & \end{tikzcd}$$

Let $l \in \mathbb{Z}S'$ then $l = j(a) - j(b)$ for $a, b \in S'$ then if

$0 = \tilde{k}(l) = \tilde{k}(j(a) - j(b)) = \tilde{k}(j(a)) - \tilde{k}(j(b)) = k(a) - k(b)$

thus $\tilde{k}$ is injective. We now show $\tilde{k}$ is surjective 3 and hence an isomorphism. Towards this we need.

• **Lemma**: Let X and Y be irreducible affine varieties. Let $\varphi: X \longrightarrow Y$ be a morphism whose image is open and dense in Y. The injection $\varphi^{\#}: k[Y] \xrightarrow{\text{inj.}} k[X]$ extends uniquely to a field isomorphism $k(Y) \xrightarrow{\tilde{\varphi}} k(X)$.

Here $k(X) = \mathrm{Frac}(k[X]) = k[X]_{\langle 0 \rangle}$
$k(Y) = \mathrm{Frac}(k[Y]) = k[Y]_{\langle 0 \rangle}$
which make sense since X and Y are irreducible so $k[X]$ and $k[Y]$ are domains.

• **Pf**: The morphism $\tilde{\varphi}: k(Y) \longrightarrow k(X)$ is from localization and give by

$\frac{f}{g} \longmapsto \frac{\varphi^{\#}(f)}{\varphi^{\#}(g)}$ this clearly extends $\varphi^{\#}$ uniquely so we must check iso.

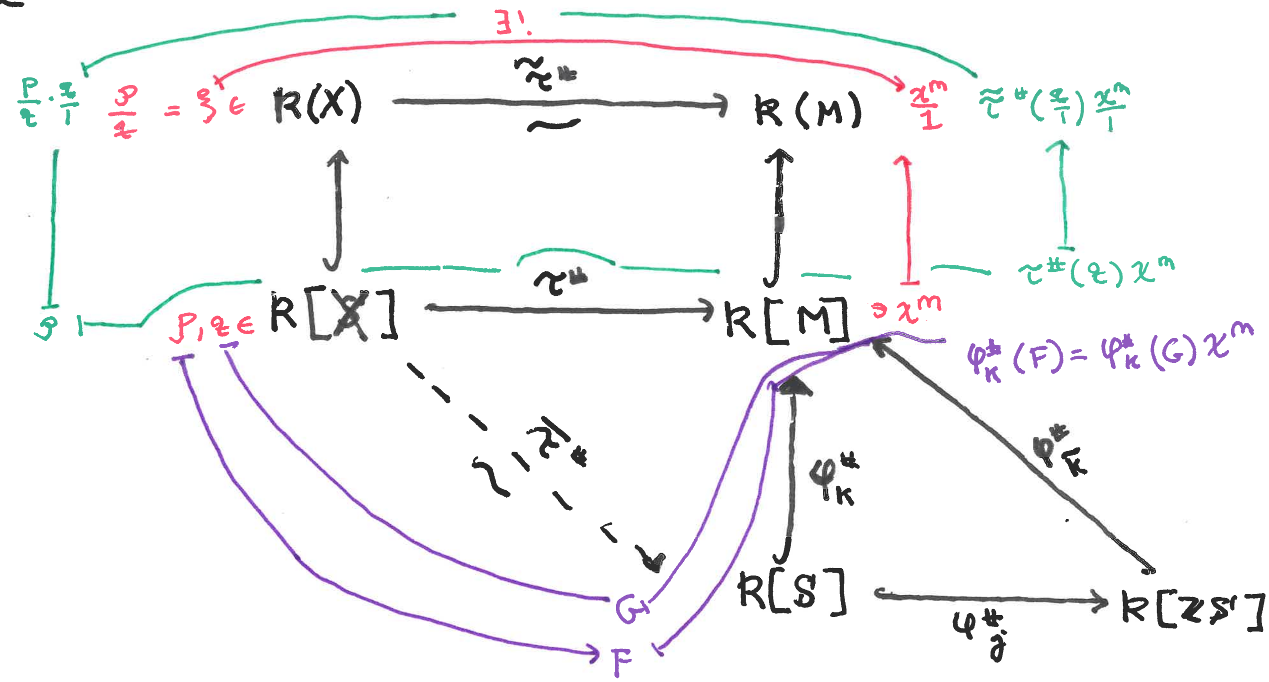
injective follows for free since $k(Y)$ is a field and $k(X) \neq 0$.

I know use the fact

$\text{tr.deg}(X)\ \dim(X) = \dim(U) = \dim(Y) = \text{tr.deg}_k(Y)$ ← this is really a pvst.

to get that $[k(X) : \tilde{\varphi}(k(Y))] = 1$

Pf: (cont.): All of this now gives the following diagram



We want to show $\tilde{\kappa}: \mathbb{Z}S \longrightarrow M$ is a bijection, which all that remains is proving surjectivity. Let $m \in M$, it is enough to show $\exists \ell \in \mathbb{Z}S'$ s.t. $\varphi^\#_{\tilde{\kappa}}(x^\ell) = x^m$. We do so by a painful diagram chase

① Consider x^m as an element $\frac{x^m}{1}$ in $k(M)$, noting the localization map is injective since $k[M]$ is a domain, since $\tilde{\tau}^\#$ is an iso. $\exists$ a unique element $\xi \in k(X)$ such that $\tilde{\tau}^\#(\xi) = \frac{x^m}{1}$. Pick $p, q \in k[x]$ such that $\xi = \frac{p}{q}$

② Now consider the image of p along $\tau^\#$ by chasing upwards

$$\frac{\tau^\#(p)}{1} = \tilde{\tau}^\#\left(\frac{p}{1}\right) = \tilde{\tau}^\#\left(\frac{q}{1}\left(\frac{p}{q}\right)\right) = \tilde{\tau}^\#\left(\frac{q}{1}\right)\tilde{\tau}^\#\left(\frac{p}{q}\right) = \tilde{\tau}^\#\left(\frac{q}{1}\right)\left(\frac{x^m}{1}\right)$$

where again we use injectivity now of the $k[x] \longrightarrow k(X)$ localization map. But the RHS and LHS are both in the image of the localization map

$$\frac{\tau^\#(p)}{1} = \frac{\tau^\#(q)}{1}\,\frac{x^m}{1} = \tau^\#(q)x^m \Rightarrow \boxed{\tau^\#(p) = \tau^\#(q)\,x^m}$$

as elements of $k[M]$!

③ Let $F = \tilde{\tau}^\#(p)$ and $G = \tilde{\tau}^\#(q)$ these are non-zero since $\tilde{\tau}^\#$ is an isomorphism the left triangle commuting $\Rightarrow$

$$\varphi^\#_\kappa(F) = \tau^\#(p) = \tau^\#(q)x^m = \varphi^\#_\kappa(G)x^m$$

• Pf : (cont): Now write

$$F = \sum_{t=1}^{n} a_t x^{s_t} \qquad G = \sum_{t'=1}^{n'} b_{t'} x^{s_t} \qquad \in k[s']$$

with the $a_t \neq 0$ and $b_{t'} \neq 0$. computing directly

$$\varphi_k^{*}(F) = \varphi_k^{*}\left(\sum_{t=1}^{n} a_t x^{s_t}\right) = \sum_{t=0}^{n} a_t x^{k(s_t)}$$

$$\varphi_k^{*}(G) = \varphi_k^{*}\left(\sum_{t'=0}^{n'} b_{t'} x^{s_{t'}}\right) = \sum_{t'=1}^{n'} b_{t'} x^{k(s_{t'})}$$

and since φ_k^{*} is injective all terms are non-zero. Thus the equality from ③

$$\varphi_k^{*}(F) = \varphi_k^{*}(G)\, x^m \Rightarrow \sum_{t=1}^{n} a_t x^{k(s_t)} = \sum_{t'=1}^{n'} b_{t'} x^{k(s_{t'})+m}$$

Some term on the RHS is non-constant and so wlog $b_1 x^{k(s_1')+m} \neq 0$

since the characters are a basis

$$a_t x^{k(s_t)} = b_1 x^{k(s_1')+m} \qquad \text{For some } t$$

$$\Rightarrow x^{k(s_t)} = x^{k(s_1')+m} \Rightarrow k(s_t) = k(s_1')+m$$

Now chosing the bottom triangle gives the stated claim.

∎

• This proof was painful and there may be a simplification but let me point out some points you seemingly cant jump past in ways you might hope to

#1 Do we really need to that τ is T_N-equivariant to get the hard inclusion $img(\tau^\#) \subseteq k[S']$ in the proof $k[S'] = img(\tau^\#)$

yes. There are open dense maps $T_N \xrightarrow{\tau} X$ that are not T_N-equivariant where $k[S'] = img(\tau^\#)$.

$$\mathbb{R}^\times \xrightarrow{\tau} \mathbb{A}^1 \qquad\qquad \mathbb{R}[x] \xrightarrow{\tau^\#} \mathbb{R}[t, t^{-1}]$$

$$t \longmapsto t + t^2 \qquad\qquad x \longmapsto t^2 + t$$

this is a surjective map. The $img(\tau^\#) \cong k[t^2+t] \neq 0$ but there are no non-constant monomials in the image of $\tau^\#$ so $S = 0$. and $k[S] = k \neq k[t^2+t]$. But this map is not equivariant for a torus action on $\mathbb{A}^1$ and this is the key.

#2 Do you really need the ~~graded~~ k-algebra isomorphism $k[X] \xrightarrow{\sim} k[S']$ to show S is finitely generated. Doesn't this follow from $k[S'] \subseteq k[M]$ and $k[M]$ being nice (i.e Noetherian) or maybe Dickson's Lemma.

yes. A k-subalgebra of a f.g. k-algebra / Noetherian ring need not ~~be~~ f.g. as a k-algebra

$$k[x,y] \supseteq k[x, xy, xy^2, xy^3, \ldots]$$

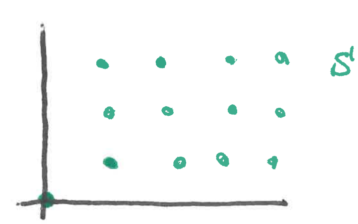
$k[x,y]$: f.g. k-algebra

$k[x, xy, xy^2, xy^3, \ldots]$: not f.g. k-algebra.

you need $B \subseteq A$ to be an integral extension for A f.g. $\Rightarrow$ B f.g. and i do not see an easier way to get that

Dickson's Lemma only applies to semigroups in $\mathbb{N}^t \subseteq \mathbb{Z}^t$, and there are semigroups in $\mathbb{Z}^t$ that are not f.g.

$$S = \{(0,0)\} \cup \{(a,b) \in \mathbb{Z}^2 \mid b \geq 0\}$$



#3 Uhh, this cannot be the easiest way to get this. We have maps of dense tori, how bad could it be?

yes and No. We actually proved the following lemma

Lemma: If $\varphi: T \longrightarrow T'$ is a map of tori that is birational then φ is an isomorphism.

Birrational ≡ isomorphism on a dense open set
≡ iso. Function fields.

I know to proofs of this fact. ① the one we used in the proof
② claim the much harder fact that a birrational homomorphism of abelian varieties is an isomorphism. Neither seems better. Further there are dense maps of Tori where the lattice map is not surjective

$$\begin{array}{ccc} \mathbb{Z} \hookrightarrow \mathbb{Z} & \quad\quad & \mathbb{R}^* \hookrightarrow \mathbb{R}^* \\ 1 \longmapsto 2 & & t \longmapsto t^2 \end{array}$$

So we are really using something.

• **Thm**: There is a bijection

$$\left\{ \begin{array}{c} \text{affine semigroups} \\ \text{upto} \\ \text{semigroup isomorphism} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{affine toric varieties} \\ \text{upto} \\ \text{equivariant isomorphism} \end{array} \right\}$$

COR: Fix a lattice M and set $N = M^\vee$, there is a bijection

$$\left\{ \begin{array}{c} \text{Finitely generated} \\ \text{semigroups} \\ S \subseteq M \\ \text{with } \mathbb{Z}S \cong M \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{Affine } T_N \\ \text{toric} \\ \text{Varieties} \end{array} \right\}$$