- Let A be a f.g. abelian group

$$D(A) = \text{"diagonalizable group scheme associated to } M\text{"} = \operatorname{Spec}(k[A])$$

$\chi^a \chi^b = \chi^{a+b}$

$\chi^0 = 1$

- <u>Thm</u>: Let A be a finitely gen. abelian group, and X an affine variety over k. Set $R = k[X]$. and $G = D(A)$.

  The following are equivalent

  1) An A-grading on R, $\left[R \cong \bigoplus_{a \in A} R_a \quad R_\alpha R_\beta \subseteq R_{\alpha+\beta}\right]$

  2) An algebraic action of G on X, and $[X \times G \longrightarrow X]$

  3) A (right) $k[A]$-coaction on R. $\left[R \xrightarrow{\ \rho\ } R \otimes k[A]\right]$

- Given a A-grading $R \cong \bigoplus R_a$

$$R \longrightarrow R \otimes k[A]$$

$$r \longmapsto r \otimes \chi^{\deg(r)} \qquad r \text{ homogeneous.}$$

- Given a $k[A]$-coaction $R \longrightarrow R \otimes k[A]$

$$R_\alpha = \{ r \in R \mid \rho(r) = r \otimes \chi^\alpha \}$$

- **Def**: A semigroup is a set S w/ a binary operation $+$ that is commutative, associative, and has an identity.

  this seems to be what others call a com. monoid.

- Given $m \in \mathbb{N} = \{0, 1, \dots\}$ and $s \in S$ a semigroup $ms = \underbrace{s + s + \cdots + s}_{m\text{-times}}$

- Given a subset $A \subseteq S$

$$\mathbb{N}(A) = \left\{ \sum_{i=1}^{n} m_i s_i \;\middle|\; \begin{matrix} m_i \in \mathbb{N} \\ s_i \in A \\ n > 0 \end{matrix} \right\} \subseteq S$$

subsemigroup generated by A.

- **Def**: A semigroup $(S, +)$ is finitely generated if there exists a finite se $A = \{a_1, \dots, a_t\} \subseteq S$ such that

$$S = \mathbb{N}(A) = \mathbb{N}a_1 + \cdots + \mathbb{N}a_t.$$

- **Def**: An affine semigroup is a finitely generated semigroup that embeds into a lattice.

- **Ex**: 1) $(\mathbb{Z}/2\mathbb{Z}, +)$ is a f.g. semigroup but does not embed into $\mathbb{Z}^n$ (torsion)

  2) $S = \{0, 1, \infty\}$ w/

	0	1	∞
0	0	1	∞
1	1	∞	∞
∞	∞	∞	∞

f.g. by 1 but (not cancelative)

$1 + 1 = 1 + \infty$

$1 \neq \infty$

so not cancelative.

- Every lattice in toric geometry will be contained in a lattice.

- If $S \subseteq M$ is an affine semigroup in a lattice M then

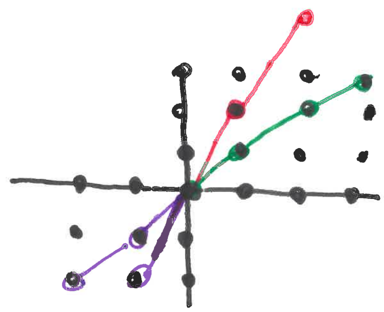
$$\mathbb{Z}S = \left\{ \sum_{i=1}^{n} m_i s_i \;\middle|\; \begin{matrix} m_i \in \mathbb{Z} \\ s_i \in S \\ n > 0 \end{matrix} \right\} \subseteq M$$

$\uparrow$ lattice generated by S

- Ex: $M = \mathbb{Z}^2$

  $\boxed{S} = \mathbb{N}(1,1)$

  $\boxed{S'} = \mathbb{N}(1,2)$



- Def: Let $S \subseteq M$ be an affine semigroup in a lattice M.
  The semigroup algebra of S is

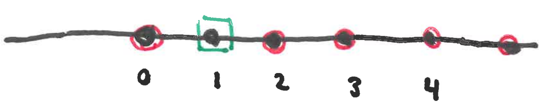
$$k[S] = \bigoplus_{m \in S} k\chi^m \subseteq k[\mathbb{Z}S] \subseteq k[M]$$

where $\chi^m \chi^{m'} = \chi^{m+m'}$ and $\chi^0 = 1$.

- Ex: $S = \mathbb{N}^n \subseteq \mathbb{Z}^n = M$

$$k[S] = k[\chi^{e_1}, \dots, \chi^{e_n}] \cong k[x_1, \dots, x_n] \subseteq k[x_1^{\pm}, \dots, x_n^{\pm}]$$

- Ex: $S = \mathbb{N}\{2,3\} \subseteq \mathbb{Z}$ $\qquad k[t^2, t^3] = k[S] \subseteq k[t, t^{-1}]$.



• **Lemma**: If S is an affine semigroup then

1) $k[S] \subseteq k[\mathbb{Z}S]$,

2) $k[S]$ is a f.g. k-algebra

3) $k[S]$ is an integral domain.

Pf: (1) is imediate/definitional.

(2) Suppose $S = \mathbb{N}\{m_1, \dots m_t\}$. Let $\chi^m \in k[S]$ be a monomial then $m = a_1 m_1 + \dots a_t m_t$

$$\chi^m = (\chi^{m_1})^{a_1} \cdots (\chi^{m_t})^{a_t}$$

$\Rightarrow k[\chi^{m_1}, \dots \chi^{m_t}] \supseteq k[S] \Rightarrow =$.

(3) $k[\mathbb{Z}S]$ is a domain $\Rightarrow k[S]$ is a subring of a domain. ∎

• Let S be an affine semigroup $k[S]$ corresponds to an affine variety X_S.

• **Def**: A topological space $X \neq \emptyset$ is irreducible iff any of the following equivalent conditions hold.

1) $X = V_1 \cup V_2$ w/ V_i closed then $V_i = X$ for some i

2) $U_1, U_2 \subseteq X$ open then $U_1 \cap U_2 \neq \emptyset$

3) every non-empty open subset is dense

• **Lemma**: An affine variety X is irreducible $\Leftrightarrow$ $k[X]$ is a domain.

Pf: Say $k[X] \cong k[x_1, \dots x_r]/\mathbb{I}(X)$. domain $\Leftrightarrow$ $\mathbb{I}(X)$ prime

$\Rightarrow$ suppose $X = \mathbb{V}(I) \cup \mathbb{V}(J)$ subsets w/ I and J radical

$$\mathbb{I}(X) = I \cap J$$

$\Leftarrow$

If $\mathbb{I}(X)$ prime then $I \subseteq \mathbb{I}(X)$ or $J \subset \mathbb{I}(V) \Rightarrow X \subseteq \mathbb{V}(I)$ or $X \subseteq \mathbb{V}(J)$

$\Rightarrow$ irreducible

$\Rightarrow$ If X is irreducible, by contrapositive

assume $\mathbb{I}(X)$ not prime and let $f, g \notin \mathbb{I}(V)$ w/ $fg \in \mathbb{I}(X)$

$$\mathbb{V}(\mathbb{I}(X) + \langle f \rangle) \cup \mathbb{V}(\mathbb{I}(X) + \langle g \rangle) = X$$

$\Rightarrow$ reducible. ∎

- **COR:** If S is an affine semigroup then X_S is irreducible.

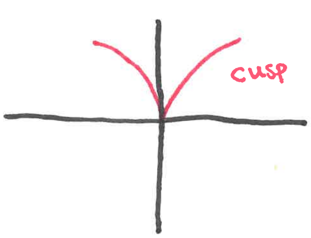
- **Lemma:** Let S be an affine semigroup. Let $A \subseteq S$ be a generating set for S $= m_1, \dots m_t$

$$k[x_1, \dots x_t] \xrightarrow{\pi_A} k[S]$$
$$x_i \longmapsto x^{m_i}$$

is a surjective ring map, i.e.,

$$X_S \cong \mathbb{V}(\ker(\pi_A)) \subseteq \mathbb{A}^t.$$

- **Ex:** $S = \mathbb{N}\{2, 3\} \subseteq \mathbb{Z}$ $\quad A = \{2, 3\}$



$$k[x, y] \xrightarrow{\pi_A} k[t^2, t^3] = k[S]$$
$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

$$\ker(\pi_A) = \langle x^3 - y^2 \rangle \subseteq k[x, y]$$

- **Question**: How can we find generators for $\ker(\pi_A)$?

- Let S be an affine semigroup $M = \mathbb{Z}S$. Note a generating set $A = \{m_1, \dots m_t\} \subseteq S$ is a generating set for M

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^t \xrightarrow{\pi_A} M \longrightarrow 0$$
$$e_i \longmapsto m_i$$

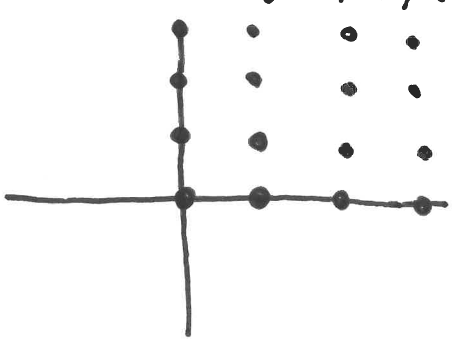
- **Def**: If $L \subseteq \mathbb{Z}^t$ is a $\overset{\text{sub}}{\text{lattice}}$ the lattice ideal

$$I_L = \langle x^u - x^v \mid u, v \in \mathbb{N}^t \text{ and } u - v \in L \rangle$$

$$= \langle x^{\ell^+} - x^{\ell^-} \mid \ell \in L \rangle \subseteq k[x_1, \dots x_t].$$

where $\ell^+ = \sum_{\ell_i > 0} \ell_i e_i \qquad \ell^- = \sum_{\ell_i < 0} -\ell_i e_i$

- **Ex**: $S = \mathbb{N}\{(2,0), (1,1), (0,2)\} \subseteq \mathbb{Z}^2$



$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix}1\\-2\\1\end{pmatrix}} \mathbb{Z}^3 \xrightarrow[\pi_A]{\begin{bmatrix}2 & 1 & 0\\0 & 1 & 2\end{bmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

$$e_1 \longmapsto (2,0)$$
$$e_2 \longmapsto (1,1)$$
$$e_3 \longmapsto (0,2)$$

$$\ker(A) = \mathbb{Z}(1,-2,1)$$

$$I_L = \langle x_1 x_3 - x_2^2 \rangle \subseteq k[x_1, x_2, x_3]$$

• **Prop**: Let S be an affine semigroup w/ $M = \mathbb{Z}S$ and $A = \{m_1, \dots m_t\}$ a generating set for S. Then

$$\begin{aligned} k[x_1, \dots x_t] &\xrightarrow{\pi_A} k[S] \\ x_i &\longmapsto \chi^{m_i} \end{aligned}$$

is a surjective morphism of k-algebras and if $L = \ker(\mathbb{Z}^t \overset{A}{\twoheadrightarrow} M)$ then

$$\operatorname{Ker}(\pi_A) = I_L$$

Pf: We show both inclusions. we have an exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^t \xrightarrow{A} M \longrightarrow 0$$

1) $I_L \subseteq \operatorname{Ker}(\pi_A)$: Let $x^u - x^v \in I_L \iff u - v \in L$ w/ $u, v \in \mathbb{N}^t$.

since $L = \operatorname{Ker}(A) \Rightarrow A(u-v) = Au - Av = 0 \Rightarrow Au = Av$

Now $\pi_A(x^u - x^v) = \chi^{Au} - \chi^{Av} = 0$.

2) $I_L \supseteq \operatorname{Ker}(\pi_A)$: Define a M-grading on $k[x_1, \dots x_t]$ by $\deg(x_i) = m_i$. The map π_A is homogeneous, and so $\operatorname{Ker}(\pi_A)$ is homogeneous in the M-grading.

Fact: $f \in I \iff f_\alpha \in I_\alpha \quad \forall \alpha$

$I \subseteq k[x_1, \dots x_t]$ an A-homog. ideal $\quad f = f_1 + \dots f_t$ with $f_\alpha \in$

• Pf: (cont.): Notice for $s \in M$ $k[x_1, \dots x_t]_s = 0$ if $s \notin S'$

and if $s \in S'$ then

$$k[x_1, \dots x_t]_s = \bigoplus_{\substack{u \in \mathbb{N}^t \\ Au = s}} k \cdot x^u$$

Let $f \in \ker(\pi_A)_s$ so $f = \sum_{Au=s} c_u x^u$ for $c_u \in k$ w/

$$0 = \pi_A(f) = \sum_{Au=s} \pi_A(x^u) c_u = \sum_{Au=s} x^{Au} c_u = \sum_{Au=s} c_u x^s = \left(\sum_{Au=s} c_u\right) x^s$$

Since x^s form a basis

$$\pi_A(f) = 0 \iff \sum_u c_u = 0$$

Now we show that given

$$f \in \ker(\pi_A)_s \iff f = \sum_{\substack{Au=s \\ i=1}}^{m} c_{u_i} x^{u_i} \quad \sum_{i=1}^{m} c_{u_i} = 0$$

then $f \in I_L$. We may write f as

$$f = \sum_{i=2}^{m} c_{u_i}(x^{u_i} - x^{u_1}) + \left(\sum_{i=1}^{m} c_i\right) x^{u_1} = \sum_{i=2}^{m} c_{u_i}(x^{u_i} - x^{u_1})$$

It is enough to show $x^{u_i} - x^{u_1} \in I_L$. but this is

clear since $Au_i = Au_1 = s$ so $u_i - u_1 \in L = \ker(A)$. ∎

• Notice since π_A is surjective

$$k[S] \cong \mathbb{C}[x_1, \dots x_t] / \ker(\pi_A)$$

$\Rightarrow \ker(\pi_A) = I_L$ is a prime ideal as $k[S]$ is a domain.

• Geometrically, π_A corresponds to a closed embedding

$$X_S \hookrightarrow \mathbb{A}^t$$

$$p \longmapsto (\chi^{m_1}(p), \dots \chi^{m_t}(p))$$

and the image of this is $\mathbb{V}(I_L)$.

• Lemma: Let $L \subseteq \mathbb{Z}^t$ be a sublattice, and I_L the associated lattice ideal.

I_L is prime $\Leftrightarrow L \subseteq \mathbb{Z}^t$ is saturated

(needs $k = \bar{k}$)

$\Leftrightarrow \mathbb{Z}^t / L$ is a lattice

Pf: $\Leftarrow$ Clear from previous argument

$\Rightarrow$ Towards a contradiction let $mv \in L$ but $v \notin L$. WLOG we may assume m is prime by saying $v' = \frac{m}{p} v$ where p is a prime $| m$.

$$(x^{v^+})^m - (x^{v^-})^m = \prod_{j=0}^{m} (x^{v^+} - \xi^j x^{v^-}) \in I_L$$

(ξ mth root of unit)
$\mathrm{Char}(k) \nmid m$.

Since $m(v^+ - v^-) = mv^+ - mv^- \in L$. $\Rightarrow x^{v^+} - \xi^j v^- \in I_L$ by primality

but $(1, \dots 1) \in \mathbb{V}(I_L) \Rightarrow 1 - \xi^j = 0 \Rightarrow \xi^j = 1 \Rightarrow m | j \Rightarrow j = 0$

$\Rightarrow x^{v^+} - x^{v^-} \in I_L$. In the remaining case use frobenius and reducedness $\blacksquare$

• **Lemma**: Let S be an affine semigroup and set $M \cong \mathbb{Z}S$.

If $\{m_1, \dots m_t\} \subseteq S$ generate S then

$$k[S]\left[\frac{1}{\chi^{m_1+\dots+m_t}}\right] \cong k[M].$$

• **Pf**: Since $k[S]$ is a domain and $k[S] \subseteq k[M]$ and $\chi^{m_1+\dots+m_t}$ is invertible in $k[M]$ we have inclusions

$$k[S] \subseteq k[S]\left[\frac{1}{\chi^{m_1+\dots+m_t}}\right] \subseteq k[M].$$

For surjectivity. let $l \in M = \mathbb{Z}S$ so we may write l as $u-v$ for $u, v \in S$ must show $\chi^{u-v} \in k[S]\left[\frac{1}{\chi^{m_1+\dots+m_t}}\right]$.

Towards this notice χ^{m_i} and χ^{-m_i} are in $k[S]\left[\frac{1}{\chi^{m_1+\dots+m_t}}\right]$. The first is clear the second is

$$\chi^{-m_i} = \frac{\chi^{m_1}\chi^{m_2}\cdots\widehat{\chi^{m_i}}\cdots\chi^{m_t}}{\chi^{m_1+\dots+m_t}}$$

But we may express u and v in the χ^{m_i} and χ^{-m_i} respectively. $\blacksquare$

• **COR**: Let $f = \chi^{m_1+\dots+m_t}$ w/ notation as above then

$\underbrace{X_S \setminus \mathbb{V}(f)}_{:= T_N}$ is an open dense torus. $N = M^\vee$.

Pf: We saw $k[T_N] \cong k[S][\frac{1}{f}] \cong k[M]$ hence T_N is a torus that is open. Density follows since X_S is irreducible. $\blacksquare$

- We want to show $T_N \hookrightarrow X_S$ in a way that stabilizes $T_N \subseteq X_S$.

- <u>Recall</u>: If $k = \bar{k}$ then Hilbert's Nullenstatz gives.

I. $\left\{ \begin{matrix} \text{Points in} \\ k^n \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{maximal ideals} \\ k[x_1, \ldots x_n] \end{matrix} \right\}$

II. $\left\{ \begin{matrix} \text{Points in} \\ k^n \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \underbrace{\frac{k[x_1, \ldots x_n]}{\mathfrak{m}}}_{\cong k} \;\middle|\; \mathfrak{m} \text{ a maximal ideal} \right\}$

III. $\left\{ \begin{matrix} \text{algebraic morphisms} \\ V \longrightarrow W \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{ring homomorphism} \\ k[W] \longrightarrow k[V] \end{matrix} \right\}$

- Thus, a point $* \in V \iff * \hookrightarrow V \iff k[V] \longrightarrow k[*]$

$\iff \underbrace{k[V] \longrightarrow k}_{\text{map of } k\text{-algebras}}$

- <u>Lemma</u>: Let M be a lattice $N = M^\vee$. There is a bijection

$$\left\{ \begin{matrix} \text{Points of} \\ T_N \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{Group Homomophisms} \\ M \longrightarrow k^\times \end{matrix} \right\}$$

- <u>Idea</u>: Elements of M are functions $T_N \longrightarrow \mathbb{G}_m$ (characters) giving a point on T_N is the same as giving a way to evaluate every function on T_N.

"Characters seperate points"

Pf: We know points are in bijection w/ k-algebra maps $\varphi : k[M] \longrightarrow k$. Given such a map define a function

$$\begin{aligned} M &\xrightarrow{\Psi_\varphi} k \\ m &\longmapsto \varphi(\chi^m) \end{aligned}$$

Notice $\Psi(m+m') = \varphi(\chi^{m+m'}) = \varphi(\chi^m)\varphi(\chi^{m'}) = \Psi(m)\cdot\Psi(m')$

Further every element of M is a unit so

$$1 = \varphi(1) = \Psi(0) = \Psi(\underset{m-m}{\cancel{}}) = \Psi(\cancel{\chi}^m)\,\Psi(m') \implies \Psi(m) \neq 0$$

so Ψ is a well-defined group homomorphism, into $k^\times$.

Conversely if $\Psi : M \longrightarrow k^\times$ is a group homomorphism define a k-algebra map

$$k[M] \longrightarrow k \quad \text{by} \quad \begin{aligned} 1 &\longmapsto 1 \\ \chi^m &\longmapsto \Psi(m) \end{aligned}$$

and extending linearly.

∎

- $T_N(k) = \mathrm{Hom}_{GRP}(M, k^\times)$

- **Lemma**: Let S be an affine semigroup $M \cong \mathbb{Z}S$ and $N = M^\vee$. There is a natural bijection

$$\begin{array}{ccc} \{\text{Pts in } T_N\} & \longleftrightarrow & \{\text{Points in } X_S\} \\ \Big\updownarrow \text{1-1} & & \Big\updownarrow \text{1-1} \\ \mathrm{Hom}_{GRP}(M, k^\times) & \longleftrightarrow & \mathrm{Hom}_{SG}(S, k). \end{array}$$

• Here $\mathrm{Hom}_{SG}(S, \mathbb{k})$ means semigroup morphisms $S' \longrightarrow \mathbb{k}$.

• Def: Let S and T be semigroups a morphism of semigroups is a function $\varphi : S \longrightarrow T$ such that

1) $\varphi(0_S) = 0_T$

2) $\varphi(s + s') = \varphi(s) + \varphi(s')$.

• Ex: . Here $\mathbb{k}$ is a semigroup under multiplication. w/ identy 1.

• Every group homomorphism is a semi-group homomorphism.

• Idea: The thing that forced the map $M \longrightarrow \mathbb{k}$ to land in $\mathbb{k}^\times$ was that every element of M is invertible. When $S \neq M$ some elements are not invertble (blow up) so land in $\mathbb{k}$ not $\mathbb{k}^\times$.

• A point $x \in X_{S'}$ is a function $\varphi_x : S' \longrightarrow \mathbb{k}$ such that

$$\varphi_x(s) = \chi^s(x) \quad \text{for all } s \in S.$$

Thus

$$\varphi_x(s) = 0 \iff \chi^s(x) = 0$$

• Maps $L \longrightarrow \mathbb{k}^\times \iff S' \longrightarrow \mathbb{k}^\times$

- **Lemma**: Let S be an affine semigroup w/ $M = \mathbb{Z}S'$ and $N = M^\vee$. The inclusion $S' \xhookrightarrow{i} M$ induces a ring map

$$k[S] \xhookrightarrow{i^{\#}} R[M]$$
$$\chi^s \longmapsto \chi^{i(s)}$$

that is injective and the induced map

$$T_N \xrightarrow{\ i\ } X_S$$

is a open embedding whose image is $D(f)$ and on points

$$i(M \xrightarrow{t} R^*) = (S \xhookrightarrow{i} M \xrightarrow{t} R^*).$$

- **Pf**: The map $i^{\#}$ is clearly injective and a ring map as it is defined on a basis. For $f = \chi^{m_1 + \cdots + m_r}$ where $m_1, \ldots, m_r$ generate S we know $i^{\#}(f)$ is a unit in $k[M]$ hence

$$\begin{tikzcd}
k[S] \arrow[r, "i^{\#}"] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowright"] & k[M] \\
k[S]_f \arrow[ur, dashed, "\cong"'] &
\end{tikzcd}$$
$$k[S]_f \cong k[M]$$

This prove $\mathrm{img}(i) \subseteq D(f)$. Surjectivity is from the --- being iso. ∎

• **Def**: Let T be a torus. An affine T-variety is an affine variety X together with an algebraic action $T \times X \xrightarrow{\Delta} X$.

• **Def**: An affine T-embedding is an affine T-variety (X, Δ) together with a T-equivariant open immersion

$$i : T \hookrightarrow X$$

whose image is dense.

equivalently an open immersion $i : T \hookrightarrow X$ whose image is dense s.t. the following commutes

$$\begin{tikzcd} T \times X \arrow[r, "\Delta"] & X \\ T \times T \arrow[u, hook, "\mathrm{id} \times i"] \arrow[r, "m"] & T \arrow[u, hook, "i"] \end{tikzcd}$$

• Note affine T-variety allows our torus to be smaller than X

• For a affine T-embedding $T \subseteq X$ being dense $\Rightarrow \dim X = \dim T$.

- Let S be an affine semigroup, $M = \mathbb{Z}S$, and $N = M^\vee$.

We saw we have an embedding

$$\begin{tikzcd} T_N \arrow[r, hook, "j"] \arrow[dr, "\sim"'] & X_S \\ & D(f) \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} k[M] & & k[S] \arrow[ll, "j^\#"'] \arrow[d] \\ & & k[S]_f \arrow[ull, "\sim"] \end{tikzcd}$$

universal property

induced by the inclusion $j : S \hookrightarrow M$.

- Want to put a T-variety structure on X_S that make i an affine T-embedding.

- There is an M-grading on $k[S]$

$$k[S]_m = \begin{cases} k \cdot \chi^s & \text{if } j(s) = m \\ 0 & \text{else} \end{cases}$$

this makes sense since $j : S \longrightarrow M$ is injective and a map of semigroups so we get the need multiplication property.

$$k[S]_m = \begin{cases} k \cdot \chi^m & m \in S \\ 0 & \text{else} \end{cases}$$

• By the theorem from previous class

$\Rightarrow$ there should be a $T_N \times X \longrightarrow X$ alg. action

$\Leftrightarrow$ Coaction $k[S] \longrightarrow k[S] \otimes k[M]$.

• Co-Action : There is a unique map making this commute

$$\begin{tikzcd}
k[S] \arrow[r, "\exists !", "\Delta"'] \arrow[d, hook, "i^{\#}"] & k[S] \otimes k[M] \arrow[d, hook, "i^{\#} \otimes \mathrm{id}"] \\
k[M] \arrow[r, "m"] & k[M] \otimes k[M]
\end{tikzcd} \qquad (*)$$

namely $\Delta(x^s) = x^s \otimes x^{j(s)}$. $[= x^s \otimes x^s]$

• Algebraic Action : On k- points

$$T_N \times X_S \longrightarrow X_S$$

$$\left([M \xrightarrow{t} k^\times], [S \xrightarrow{x} k]\right) \longmapsto \left(\left[\begin{array}{l} S \longrightarrow k \\ s \longmapsto t(j(s)) \cdot x(s) \end{array}\right]\right)$$

$$\left[= \begin{array}{l} S \longrightarrow k \\ s \longmapsto t(s)\, x(s) \end{array}\right]$$

• (*) actually shows this is an affine T- embedding.

• **Prop**: Let S be an affine semigroup and

$$S \xhookrightarrow{j} \mathbb{Z}S = M,$$

set $N = M^\vee$. There is a unique ~~T~~affine T_N-variety structure on X_S, given by

$$\begin{aligned} k[S] &\xrightarrow{\Delta} k[M] \otimes k[S] \\ \chi^{s} &\longmapsto \chi^{j(s)} \otimes \chi^{s} \end{aligned}$$

such that the inclusion induced by j:

$$T_N \xhookrightarrow{i} X_S$$

is an affine T_N-embedding.

Pf: $\blacksquare$

• Note there are two comps:

① A toric variety $\equiv$ affine T-variety + affine T-embedding $\star$

② A toric variety $\equiv$ normal affine T-variety + affine T-embedding

• We will mostly be in comp ① so we have just proved.

• **Cor**: If S is an affine semi-group X_S is a toric variety.