• **Def**: A group variety is a variety G together w/ morphisms

$$m : G \times G \longrightarrow G, \quad i : G \longrightarrow G, \quad e : * \hookrightarrow G \quad (*)$$

which satisfy the normal group ~~operati~~ axioms

• **Ex**: $\mathbb{G}_m^r = (k^\times)^r = \mathbb{A}^r \setminus \mathbb{V}(x_0 \cdots x_r)$

• **Prop**: An affine variety G is a group variety $\Longleftrightarrow$ $k[G]$ f.g. commutative reduced Hopf algebra over k

• The maps of $(*)$ ~~satis~~ induce maps of k-algebras

$$m^\# = \Delta : k[G] \longrightarrow k[G \times G] \cong k[G] \otimes_k k[G]$$

$$i^\# : S : k[G] \longrightarrow k[G]$$

$$e^\# \ \varepsilon : k[G] \longrightarrow k$$

these satisfy the dual group axioms

• Note Δ is very concrete

$$\Delta(f) = m^\#(f) = f \circ m : \left\{ \begin{array}{l} G \times G \xrightarrow{m} G \xrightarrow{f} k \\ (x,y) \longmapsto xy \longmapsto f(xy) \end{array} \right.$$

- An algebraic torus is a variety such that $T \cong \mathbb{G}_m^n$ some k

  + Not true over a general field
  + Necessarily affine
  + $k[T] \cong k[\mathbb{G}_m^n] \cong k[x_1, \dots x_n][(x_1 \cdots x_n)^{-1}]$

$$\cong k[x_1^{\pm}, \dots, x_n^{\pm}] \quad \text{"Laurent Monomials"}$$

- Lemma: $(k[x_1^{\pm}, \dots x_n^{\pm}])^{\times} = \{c\, x_1^{m_1} \cdots x_n^{m_n} \mid c \in k^{\times} \text{ and } m_i \in \mathbb{Z}\}$

- Def: A character of a group variety G is a morphism of ~~alg~~. group variete

$$G \longrightarrow \mathbb{G}_m$$

Note $\mathbb{G}_m = GL_1(k)$

So a character $\equiv$ 1-dim representation

- Lemma: Every character $\chi : \mathbb{G}_m^n \longrightarrow \mathbb{G}_m$ is of the form

$\chi(t_1, \dots t_n) = t_1^{m_1} \cdots t_n^{m_n}$ for some fixed $(m_1, \dots m_n) \in \mathbb{Z}^n$

and every $(m_1, \dots m_n)$ appears

$$\mathrm{Hom}_{GrpVar}(\mathbb{G}_m^n, \mathbb{G}_m) \cong \mathbb{Z}^n$$

Pf: A χ corresponds to $\chi^{\#} : k[t, t^{-1}] \longrightarrow k[x_1^{\pm}, \dots x_n^{\pm}]$

a k-alg morphism. $\chi^{\#}$ is determined by $\chi^{\#}(t)$, which

must be a unit so $\chi^{\#}(t) = c\, x_1^{m_1} \cdots x_n^{m_n}$ $c \in k^{\times}$ and $m_1, \dots m_n \in \mathbb{Z}$

Pf: (cont.) We now show $c = 1$. Let $\chi^{a}(t) = f \in k[G]$

$$\Delta(f)(g,h) = f(gh) \overset{(*)}{=} \chi(gh) = \chi(g)\chi(h)$$

$$\Rightarrow f(gh) = f(g)f(h)$$

$$\Rightarrow f : G \longrightarrow k \text{ is a group hom.}$$

$$\Delta(f) = f \otimes f$$

Returning to $f = c x_1^{m_1} \cdots x_n^{m_n}$ since $c \in k^\times$ and Δ is k-algebra map

$$\Delta(f) = \Delta(c x_1^{m_1} \cdots x_n^{m_n})$$

$$= c\Delta(x_1^{m_1} \cdots x_n^{m_n}) = c\,(x_1^{m_1} \cdots x_n^{m_n} \otimes x_1^{m_1} \cdots x_n^{m_n})$$

But also

$$\Delta(c x_1^{m_1} \cdots x_n^{m_n}) = (c x_1^{m_1} \cdots x_n^{m_n}) \otimes (c x_1^{m_1} \cdots x_n^{m_n})$$

$$= c^2 (x_1^{m_1} \cdots x_n^{m_n} \otimes x_1^{m_1} \cdots x_n^{m_n})$$

So $c^2 = c$ and $c = 1$. ∎

Cor: If $f : \mathbb{G}_m^n \longrightarrow \mathbb{G}_m^r$ is a group homomorphism then

$$f(t_1, \ldots, t_n) = \left(\prod_{i=1}^{n} t_i^{a_{1i}}, \ldots, \prod_{i=1}^{n} t_i^{a_{ri}}\right)$$

Moreover

$$\mathrm{Hom}_{\mathrm{GrpVar}}(\mathbb{G}_m^n, \mathbb{G}_m^r) \cong M_{r\times n}(\mathbb{Z}).$$

maps of tori are monomial maps.

Pf: Composing w/ the i^{th} projection gives a character, apply prop. $\blacksquare$

- **Def**: Let T be an alg. torus its character lattice is

$$X^*(T) = \mathrm{Hom}_{\mathrm{GrpVar}}(T, \mathbb{G}_m)$$

By previous work

+ $X^*(T) \cong \mathbb{Z}^n$ so lattice of rank n
+ we generally think of $X^*(T)$ w/ multiplication

★ Notation: $M = X^*(T)$ = character lattice

= "regular functions on T"

- What we saw for $\mathbb{G}_m^n$

1) $R[\mathbb{G}_m^n] \cong R[x_1^{\pm}, \ldots x_n^{\pm}]$

2) $\mathrm{Hom}_{\mathrm{GrpVar}}(\mathbb{G}_m^n, \mathbb{G}_m) \cong \{\text{Laurent Monomials}\} \cong \mathbb{Z}^n$

$$\Rightarrow R[\mathbb{G}_m^n] \cong R[\text{Laurent Monomials}] \cong \bigoplus_{\vec{m} \in \mathbb{Z}^n} R x^{\vec{m}}$$

- All regular functions are characters & we have a direct sum decomp.

$$f: \mathbb{G}_m^n \longrightarrow \mathbb{G}_m \subseteq \mathbb{A}^1 = R$$

• Def: If A is a f.g. abelian group its group algebra

$$k[A] \cong \bigoplus_{a \in A} k\chi^a$$

where multiplication is $\chi^a \cdot \chi^b = \chi^{a+b}$.

• Ex: $A = \mathbb{Z}^n$ then $k[\mathbb{Z}^n] \cong k[x_1^{\pm}, \dots, x_n^{\pm}]$ with a decomp.

• Thm: Let T be an n-dim torus. Fix $m_1, \dots, m_n \in M$ be a basis for M.

① Viewing $m_1, \dots, m_n$ as regular functions

$$T \xrightarrow{(m_1, \dots, m_n)} \mathbb{G}_m^n$$

is an iso. of group varieties.

② There is an isomorphism of Hopf k-algebras

$$\begin{aligned} k[M] &\xrightarrow{\sim} k[T] \\ \chi^m &\longmapsto m \end{aligned}$$

* Motto: A torus is recovered from its character lattice *

$$k[T] \cong k[M] \cong \bigoplus_{m \in M} k\chi^m$$

• COR: $(k[T])^\times \cong \{c\chi^m \mid c \in k^\times, m \in M\}$

characters are invertible on the torus

PF: Let $e_1, \dots, e_n$ be the standard basis for $X^*(\mathbb{G}_m^n) \cong \mathbb{Z}^n$ where

$$e_i(t_1, \dots, t_n) = t_i .$$

Since T is a torus, we may fix $\alpha : T \longrightarrow \mathbb{G}_m^n$, pull back induces an iso. $\alpha^\#: R[\mathbb{G}_m^n] \xrightarrow{\sim} R[T]$, which induces

$$\mathbb{Z}^n \cong X^*(\mathbb{G}_m^n) \xrightarrow[\alpha^\#]{\sim} X^*(T) \cong M$$

Define $u_i := (\alpha^\#)^{-1}(m_i) \in X^*(\mathbb{G}_m^n)$. Since the m_i are a basis and $\alpha^\#$ an iso $\Rightarrow u_1, \dots u_n$ is a basis for $\mathbb{Z}^n$

Therefore, there exists a CoB matrix $A \in GL_n(\mathbb{Z})$ taking $e_1, \dots e_n$ to $u_1, \dots u_n$.

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, "A", "\sim"'] \arrow[d, phantom, "\cong" description] & \mathbb{Z}^n \arrow[d, phantom, "\cong" description] \\
X^*(\mathbb{G}_m^n) \arrow[r, "\sim"'] & X^*(\mathbb{G}_m)
\end{tikzcd}$$

This gives a monomial automorphism

$$\mathbb{G}_m^n \xrightarrow{\mu_A} \mathbb{G}_m^n$$

w/ the property that $\mu_A^\#(e_i) = u_i$. Now consider

$$\begin{tikzcd}
T \arrow[r, "\sim"'] \arrow[dr, "\Phi"'] & \mathbb{G}_m^n \arrow[d, "\mu_A"] \\
& \mathbb{G}_m^n
\end{tikzcd}$$

we claim this diagram commutes.

Pf: (cont): It is enough to check on rings

$$\begin{tikzcd} k[T] & k[\mathbb{G}_m^n] \arrow[l, "\alpha^*"', "\sim"] \\ & k[\mathbb{G}_m^n] \arrow[u, "\mu_A^\#"'] \arrow[ul, "\Phi^\#"] \end{tikzcd}$$

which we may do on the e_i as they generated $k[\mathbb{G}_m^n]$

$$(\mu_A \circ \alpha)^\#(e_i) = \alpha^\#(\mu_A^*(e_i))$$
$$= \alpha^\#(u_i) = m_i$$

This finishes the first part.

For the second part, one check the map is a homomorphism of k-algebras directly. A similar argument shows the basis for M induces an iso

$$k[x_1^\pm, \dots x_n^\pm] \xrightarrow{\sim} k[M]$$
$$x_i \longmapsto \chi^{m_i}$$

But then $\Phi^\#: k[\mathbb{G}_m^n] \xrightarrow{\sim} k[T]$ is also an iso by (1)

So composing gives the claim.

∎

• Def: A one-parameter subgroup of a torus T is a grp. var. hom.

$$\mathbb{G}_m \xrightarrow{\lambda} T$$

• **Def**: If T is a torus

$$X_*(T) = \mathrm{Hom}_{\mathrm{GrpVar}}(\mathbb{G}_m, T)$$

we call this the <u>cocharacter lattice</u> of T.

★ **Notation**: $N = X_*(T)$ = cocharacter lattice

= "paths/curves which will control limits"

• **Lemma**: $\mathrm{Hom}_{\mathrm{GrpVar}}(\mathbb{G}_m, \mathbb{G}_m^n) \cong \mathbb{Z}^n$

Pf: This follows from our description of tori maps ∎

• **Cor**: $N := X_*(T)$ is a rank n lattice. ∎

• For $u \in N$ write λ_u: for the corresponding map $\lambda_u : \mathbb{G}_m \longrightarrow T$.

• Fix $m \in M$ and $u \in N$ so we have morphisms

$$\chi^m : T \longrightarrow \mathbb{G}_m \quad \& \quad \lambda_u : \mathbb{G}_m \longrightarrow T$$

the composition

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "\lambda_u"] \arrow[rr, bend right, "t \longmapsto t^a"'] & T \arrow[r, "\chi^m"] & \mathbb{G}_m \end{tikzcd}$$

must be of the form $t \longmapsto t^a$ for a unique $a \in \mathbb{Z}$.

• **Def**: Fix an n-dim torus T and $M = X^*(T)$ and $N = X_*(T)$.

Define a pairing

$$\langle , \rangle : M \times N \longrightarrow \mathbb{Z}$$

By $\langle m, u\rangle := a \in \mathbb{Z}$ such that $\chi^m \circ \lambda_u(t) = t^a$

i.e., $\chi^m(\lambda_u(t)) = t^{\langle m,u\rangle}$.

• **Thm**: With T, M, N as in the above definition. The pairing $\langle -, -\rangle$ is perfect, equivalently,

$$N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \quad ; \quad M \cong \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}).$$

• **Ex**: $T = \mathbb{G}_m^n$ $\quad m = (m_1, \ldots, m_n) \in M \cong \mathbb{Z}^n$

$u = (u_1, \ldots, u_n) \in N \in \mathbb{Z}^n$

$$\begin{array}{ccccc} \mathbb{G}_m & \xrightarrow{\lambda_u} & \mathbb{G}_m^n & \xrightarrow{\chi^m} & \mathbb{G}_m^1 \\ t & \longmapsto & (t^{u_1}, t^{u_2}, \ldots, t^{u_n}) & \longmapsto & t_0^{u_1 m_1} \cdots t^{u_n m_n} \\ & & (t_1, \ldots, t_n) & \longmapsto & t_1^{m_1} \cdots t_n^{m_n} \end{array}$$

$\Rightarrow \chi^m \circ \lambda_u = [t \longmapsto t^{m \cdot u}]$

$= \langle m, u \rangle =$ std. inner product.

Pf: Reduce to the $\mathbb{G}_m^n$ case and use example ∎

- *Idea*: Toric geometry has two dual lattices M & N.

- Notice we have constructed a map

$$\begin{array}{ccc} \{\text{tori}\} & & \{\text{lattices}\} \\ T & \longmapsto & X^*(T) \end{array}$$

- Let N be a lattice we want a torus T_N

$$M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$$

then $k[M]$ is a f.g. reduced k-algebra so there is some variety, call it T_N s.t. $k[T_N] = k[M]$.

Moreover since $k[M] \cong k[x_1^{\pm}, \ldots, x_n^{\pm}]$ we know T_N is a torus.

- *Thm*: Let N and N' be lattices. If $\varphi : N \longrightarrow N'$ is a lattice map then there is a torus homomorphism

$$f_\varphi : T_N \longrightarrow T_{N'}$$

given on coordinate rings by

$$\begin{array}{cccc} f_\varphi^\# : & k[T_{N'}] & \longrightarrow & k[T_N] \\ & \chi^{m'} & \longmapsto & \chi^{\varphi^\vee(m')} \end{array}$$

where $\varphi^\vee$ is the induced maps on dual lattices $\varphi^\vee : M' \longrightarrow M$.

Moreover, all torus morphisms arrive this way.

$$\mathrm{Hom}_{\text{GrpVar}}(T_N, T_{N'}) \cong \mathrm{Hom}(N, N')$$
$$\cong \mathrm{Hom}(M', M).$$

• **Thm**: There is an equivalence of categories

$$\mathrm{Lat} \cong \mathrm{Tor}_k$$

$$\begin{array}{ccccccc} \mathrm{Lat} & \xrightarrow{\vee} & \mathrm{Lat} & \xrightarrow{k[\ ]} & k\text{-alg} & \longrightarrow & \text{tori} \\ N & \longmapsto & M & \longrightarrow & k[M] & \longrightarrow & T_N \end{array}$$

• **Ex**: Consider $\mathbb{Z} \xhookrightarrow{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} \mathbb{Z}^2$ this is a lattice map

the dual map is $\mathbb{Z}^2 \xrightarrow{[\,2\ \ 1\,]} \mathbb{Z}$ by Theorem

$$\begin{array}{ccc} k[T_{\mathbb{Z}^2}] & \longrightarrow & k[T_{\mathbb{Z}}] \\ x_1\ \ x_2 & \longmapsto & t^2\ \ t \end{array}$$

• **Prop**: (SNF for Tori): Let $\varphi : N \longrightarrow N'$ be a map of lattices and $f_\varphi : T_N \longrightarrow T_{N'}$ the induced map. There exists a basis of N and N' s.t.

$$\varphi(e_i) = \begin{cases} d_i e_i' & 1 \le i \le r \\ 0 & \text{else} \end{cases}$$

where $r = \operatorname{rank}(\varphi)$ and so up to automorphism

$$f_\varphi(t_1, \dots t_n) = (t_1^{d_1}, \dots\dots t_r^{d_r}, t_{r+1}^0, \dots t_m^0)$$
$$= (t_1^{d_1}, \dots\dots t_r^{d_r}, 1, \dots 1).$$

- **Cor**: The image of a torus hom. is a sub tori.

PF: Since $\mathbb{k}$ is alg. closed the d_i-th powers dont matter

$$img(f_\varphi) = \{(y_1, \dots, y_m) \in T_{N'} \mid y_{r+1} = \dots = y_m = 1\}$$

$\boxtimes$

- **Ex**: Consider the lattice map

$$\varphi: \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{SNF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f_\varphi = (x_1, x_2, x_3) \longmapsto (x_1, x_2^2, x_3^2, 1)$$

- **Cor**: For $f_\varphi : T_N \longrightarrow T_{N'}$ FFAE

1) f_φ is surjective
2) $rk(\varphi) = rk\ N'$
3) $Coker(\varphi)$ finite
4) $\varphi^\vee$ injective

1) f_φ closed embedding
2) $\varphi^\vee$ surjective
3) φ injective $\ni$ $\varphi(N) \subseteq N'$ saturated

- **Def**: Let $L' \subseteq L$ be a sublattice in L then L' is saturated in $L \iff mv \in L'$ for $m > 0$ then $v \in L'$.

- Lemma: $L' \subseteq L$ saturated $\iff$ L/L' torsion free.

- Pf: (Cor): For the RHS immersion $\Rightarrow$ $d_i = 1$ $\iff$ coker torsion free.

- Lemma: Let G be a affine ~~alg.~~ group variety. If $\chi_1, \ldots \chi_r$ are distinct characters $G \longrightarrow \mathbb{G}_m$ then $\chi_1, \ldots \chi_r$ are linearly ind. $/k$ as elements of $k[G]$.

Pf: Suppose not, and choose a minimal length relation

$$\Psi = a_1\chi_1 + \cdots a_t\chi_t = 0 \qquad a_i \in k^\times$$

Pick $g \in G$ we get a new character

$$G \xrightarrow{g \cdot} G \xrightarrow{\Psi} \mathbb{G}_m$$
$$h \longmapsto gh \longmapsto \Psi(gh)$$

$$\Psi(gh) = (a_1\chi_1 + \cdots a_t\chi_t)(gh)$$

$$= a_1\chi_1(gh) + \cdots a_t\chi_t(gh)$$

$$= a_1\chi_1(g)\chi(h) + \cdots\cdots + a_t\chi_t(g)\chi_t(h) = 0$$

$$\Rightarrow a_1\chi_1(g)\chi_1(-) + \cdots\cdots a_t\chi_t(g)\chi_t(-) = 0 \quad \text{(as a character)}$$

Subtracting $\chi_1(g)\Psi = 0$

$$(a_1\chi_1(g)\chi_1(-) + \cdots a_t\chi_t(g)\chi_t(-)) - \chi_1(g)\Psi = 0$$

$$= a_2(\chi_2(g) - \chi_1(g))\chi_2 + \cdots a_t(\chi_t(g) - \chi_1(g))\chi_t = 0 \ \#$$

- Idea: χ_i is egenvector for tralslation by g w/ eigenvalue $\chi_i(g)$.

- Def: Let N be a lattice with dual M so $k[T_N] \cong k[M]$.

  Let $H \subseteq T_N$ be a closed group subvariety. Define

$$L_H = \{ m \in M \mid \chi^m|_H = 1 \} \qquad I_H = \langle \chi^\ell - 1 \mid \ell \in L_H \rangle$$

- Thm: With notation as above. If $H \subseteq T_N$ is a closed group subvarety then

  1) $\mathbb{I}(H) = I_H$

  2) $k[H] \cong k[T_N]/I_H \cong k[M/L_H]$.

- Ex: $N = \mathbb{Z}^2$ $\quad k[\mathbb{G}_m^2] \cong k[x^\pm, y^\pm] \qquad x = \chi^{(1,0)} \quad y = \chi^{(0,1)}$

  $+ \; L = \mathbb{Z} \cdot (1,1) \subseteq \mathbb{Z}^2$

  $I_L \langle \chi^\ell - 1 \mid \ell \in L \rangle = \langle \chi^{m(1,1)} - 1 \mid m \in \mathbb{Z} \rangle = \langle xy - 1 \rangle$

  $H = \{ (x,y) \in (k^*)^2 \mid xy = 1 \}$ torus

$$k[H] = \frac{k[x^\pm, y^\pm]}{\langle xy - 1 \rangle} \cong k[x^\pm]$$

• **Ex**: $L = \mathbb{Z}(2,0) \subseteq \mathbb{Z}^2$

$I_L = \langle x^2 - 1 \rangle$

$H = \{ (x,y) \in (k^\times)^2 \mid x^2 = 1 \}$

$= \{ (1,y) \mid y \in k^\times \} \cup \{ (-1,y) \mid y \in k^\times \} = \mu_2 \times \mathbb{G}_m$

Note L is not saturated as $\mathbb{Z}^2/L \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$

or $(2,0) \in L$ but $(1,0) \notin L$ and $2 \cdot (1,0) = (2,0)$.

Pf: The inclusion $I_H \subseteq \mathbb{I}(H)$ is clear.

Take $f = \sum_m a_m \chi^m \in \mathbb{I}(H)$ WTS $f \in I_H$ i.e. $f|_H = 0$.

* **Aside**: suppose $\chi^m, \chi^{m'} \in R[T]$ s.t. $\chi^m|_H = \chi^{m'}|_H$

$$\chi^m|_H = \chi^{m'}\chi^{m-m'}|_H = \chi^{m'}|_H \, \chi^{m-m'}|_H = \chi^{m'}|_H$$

Thus, $\chi^m|_H = \chi^{m'}|_H \iff \chi^{m-m'}|_H = 1 \iff m - m' \in L$.

Let $\mathcal{C}$ be a set of cosets for M/L and for each m choose a rep. m_c so we write

$$f|_H = \sum_{c \in \mathcal{C}} \Big(\sum_{m \in c} a_m \Big) \chi^{m_c}|_H$$

Since c are distinct cosets $\chi^{m_c} \neq \chi^{m_{c'}}$ on H.

But now $f|_H = 0$ but χ^{m_c} are linearly ind. by lemma

Pf: (cont): $\Rightarrow \sum_{m \in c} a_m = 0$ for all $c \in \mathcal{C}$.

$$\Rightarrow \sum_{m \in c} a_m \chi^{m_c} = 0 \quad \text{on } T$$

$$\Rightarrow \sum_{m \in c} a_m \chi^m - \sum_{m \in c} a_m \chi^{m_c} = \sum_{m \in c} a_m \chi^m$$

$$= \sum_{m \in c} a_m (\chi^m - \chi^{m_c})$$

$$\Rightarrow f = \sum_{c \in \mathcal{C}} \sum_{m \in c} a_m (\chi^m - \chi^{m_c})$$

We now will show $(\chi^m - \chi^{m_c}) \in I_H$ for m and m_c in some coset (else not in sum): but $m, m_c \in C$ $\Leftrightarrow m - m_c \in L \Rightarrow \chi^m - \chi^{m_c} = \chi^{m_c}\underbrace{(\chi^{m-m_c} - 1)}_{\cap \, I_H}$.

So $f \in \mathbb{I}(H)$.

The second claim follows form

$$k[M]/\langle \chi^\ell - 1, \ell \in L \rangle \Leftrightarrow \chi^m = \chi^{m+\ell} \quad \forall \ell \in L$$

• COR: If $H \subseteq T$ is a connected group subvariety $\Rightarrow H$ is a torus.

Pf: Since $H \supseteq H$ connected $\Rightarrow$ irreducible

$\Rightarrow k[H]$ is a domain

$\Rightarrow k[M/L]$ is a domain

$\Rightarrow L \subseteq M$ a lattice ∎

- **Lemma**: If $N' \subseteq N$ is a saturated sublattice

$$T_N / T_{N'} \cong T_{N/N'}$$

induced by the natural qoutient map $\pi : N \to N/N'$.

- **Ex**: $N = \mathbb{Z}^{n+1}$ $\quad \mathbb{G}_m^{m+1}$ $\quad \mathbb{G}_m^{m+1} / \Delta \mathbb{G}_m \cong T_{\mathbb{Z}^{n+1}/\langle 1, \dots, 1 \rangle}$

$\cup\!\!\mid$ $\quad \cup\!\!\mid$

$N' = \mathbb{Z}(1,1,\dots,1)$ $\quad \Delta \mathbb{G}_m$ $\quad \cong \mathbb{G}_m^n$.

- An algebraic torus is a group so we can discuss rep theory!

- **Def**: A representation of T is a group hom.

$$\rho : T \longrightarrow GL(V)$$

for a k-vector space V. $\Leftrightarrow$ $T \times V \longrightarrow V$ $(*)$

An action is algebraic if $(*)$ is a morphism

For $m \in M$ the m-weight space is

$$V_m := \{ v \in V \mid t \cdot v = \chi^m(t) v \quad \forall t \in T \}$$

- **Ex**: $T = \mathbb{G}_m$, $V = k^2$

$$t \cdot (x, y) = (tx, t^2 y)$$

$(1,0) \in V_1$ $\quad (0,1) \in V_2$ $\quad (1,1) \notin V_m$

• Thm : If $T \times V \longrightarrow V$ is a f.d. algebraic representation of T then

$$V = \bigoplus_{m \in M} V_m .$$

every rep is diagonalizable

• Ex : $\rho(t) = \begin{pmatrix} \frac{t+t^2}{2} & \frac{t-t^2}{2} \\ \frac{t-t^2}{2} & \frac{t+t^2}{2} \end{pmatrix}$ gives a rep of $\mathbb{G}_m$ $V \cong \mathbb{R}^2$

$V = V_1 \oplus V_2$ $\quad V_1 = \text{span} \langle f_1 + f_2 \rangle$

$V_2 = \text{span} \langle f_1 - f_2 \rangle$

• We can generalize the above Thm.

$T = T_N \qquad M = X^*(T)$

Let X be an affine variety and $T \times X \xrightarrow{\alpha} X$ an algebraic action

For notation $A = k[X]$.

$$\alpha^\#: A \longrightarrow k[T \times X] \cong k[M] \otimes_k M$$

$$A_m = \{ a \in A \mid \alpha^\#(a) = \chi^m \otimes a \}$$

• Thm: With the set-up as before.

$$A \cong \bigoplus_{m \in M} A_m$$

Such that $A_m \cdot A_{m'} \subseteq A_{m+m'}$ and $A^T = A_0$, i.e. M-grading on A.

• Ex: $T = \mathbb{G}_m^n \curvearrowright \mathbb{A}^n$

$$\begin{array}{ccc} ((t_1, \dots, t_n), (x_1, \dots, x_n)) & \longrightarrow & (t_1 x_1, \dots, t_n x_n) \\ T \times \mathbb{A}^n & \xrightarrow{\alpha} & \mathbb{A}^n \end{array}$$

$$k[x_1, \dots, x_n] \xmapsto{\alpha^\#} k[T] \otimes_k k[x_1, \dots, x_n]$$

$$\cong$$

$$k[t_1^{\pm}, \dots, t_n^{\pm}] \otimes k[x_1, \dots, x_n]$$

$$x_i \longmapsto t_i \otimes x_i$$

$\Rightarrow x_i \in A_{t_i}$ which $k[T] \cong k[M] \cong k[\mathbb{Z}^n]$

$\Rightarrow \deg(x_i) = e_i \in \mathbb{Z}^n$.

• Ex: $X = \mathbb{V}(xy - z^2) \subseteq \mathbb{A}^3$ $\quad T = \mathbb{G}_m$

$$\begin{array}{ccc} T \times X & \xrightarrow{\alpha} & X \\ (t, (x, y, z)) & \longmapsto & (tx, t^{-1}y, z) \end{array}$$

$(tx)(t^{-1}y) - z^2$
$= xy - z^2$ ✓

$$\frac{k[x,y,z]}{\langle xy - z^2 \rangle} \xrightarrow{\alpha^\#} k[t, t^{-1}] \otimes \frac{k[x,y,z]}{\langle xy - z^2 \rangle}$$

$x \longmapsto t \otimes x$ $\qquad z \longmapsto 1 \otimes z$.

$y \longmapsto t^{-1} \otimes y$

• Ex: (cont): $\deg(x) = 1$, $\deg(y) = -1$, $\deg(z) = 0$