• Let k be a field

• <u>Def</u>: An <u>algebraic set</u> is a set of the form:

$$\mathbb{V}(f_1, \dots, f_t) = \{ \bar{x} \in \mathbb{A}^n_k \mid f_1(\bar{x}) = \dots = f_t(\bar{x}) = 0 \} \subseteq \mathbb{A}^n_k$$

More generally, given an ideal $I \subseteq k[x_1, \dots, x_n]$

$$\mathbb{V}(I) = \{ \bar{x} \in \mathbb{A}^n_k \mid f(\bar{x}) = 0 \ \forall f \in I \}$$

• <u>Remarks</u>:

1) If $I = \langle f_1, \dots, f_t \rangle$ then $\mathbb{V}(I) = \mathbb{V}(f_1, \dots, f_t)$

2) Hilbert's Basis Theorem $\Rightarrow$ every ideal in $k[x_1, \dots, x_n]$ is of this form.

• <u>Def</u> : Given an algebraic set $V \subseteq \mathbb{A}^n_k$

$$\mathbb{I}(V) = \{ f \in \mathbb{C}[x_1, \dots, x_n] \mid f(a) = 0 \ \forall a \in V \}$$

• <u>Claim</u>: $\mathbb{I}(V)$ is an ideal.

• <u>Lemma</u>: If $V \subseteq \mathbb{A}^n_k$ is an alg. set

$$V = \mathbb{V}(\mathbb{I}(V))$$

• <u>Prop</u>: Assume $k = \bar{k}$. ~~then~~ If $I \subseteq k[x_1, \dots, x_n]$ is an ideal

$$\mathbb{I}(\mathbb{V}(I)) = \{ f \in k[x_1, \dots, x_n] \mid f^k \in I \text{ some } k \}$$

$$:= \sqrt{I}$$

Thm: (Nullstellensatz I): Assume $k = \bar{k}$.

① An ideal $\mathfrak{m} \subseteq k[x_1, \dots, x_n]$ is maximal $\iff \mathfrak{m} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$ for some $(a_1, \dots, a_n) \in k^n$.

② There is an inclusion reversersing bijection

$$\left\{ \begin{matrix} \text{algebraic subsets} \\ \mathbb{A}^n_k \end{matrix} \right\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \left\{ \begin{matrix} \text{radical ideals} \\ \text{in } \mathbb{C}[x_1, \dots, x_n] \end{matrix} \right\}$$

- **Ex**: $k = \mathbb{R}$
  - $\langle x^2+1 \rangle \subseteq \mathbb{R}[x]$ is maximal $\langle x^2+1 \rangle \neq \langle x-a \rangle$ for any $a \in \mathbb{R}$
  - $\mathbb{V}(\langle x^2+1 \rangle) = \emptyset$, $\mathbb{I}(\emptyset) = \mathbb{R}[x] \Rightarrow \mathbb{I}(\mathbb{V}(\langle x^2+1 \rangle)) = \mathbb{R}[x]$

$$\neq \langle x^2+1 \rangle = \sqrt{\langle x^2+1 \rangle}$$

- Given a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ there is a function

$$k^n \xrightarrow{f} k \qquad \bar{x} \longmapsto f(\bar{x})$$

Functions of this form are called polynomial functions on $\mathbb{A}^n_k$.

- **Def**: A polynomial function on an algebraic set $V \subseteq \mathbb{A}^n_k$ is a function $f : V \longrightarrow k$ such that there exists a polynomial function $F : \mathbb{A}^n_k \longrightarrow k$ w/

$$F|_V = f$$

• **Def**: Given an algebraic set $V \subseteq \mathbb{A}^n_{\mathbb{R}}$ the coordinate ring of V is the ring

$$\mathbb{R}[V] = \{ V \xrightarrow{f} \mathbb{R} \mid f \text{ is a polynomial function on } V \}$$

under pointwise addition and multiplication.

• **Prop**: ① $\mathbb{R}[\mathbb{A}^n_{\mathbb{R}}] \cong \mathbb{R}[x_1, \ldots, x_n]$

② $\mathbb{R}[V] \cong \mathbb{R}[x_1, \ldots, x_n]/\mathbb{I}(V)$.

• **Ex**: + Sometimes two polynomials define the same function

$V = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2_{\mathbb{R}}$

$f = y$

$$g = 2y - x^2$$
$$= y + (y - x^2) = y + 0$$

If $(x,y) \in V$ then $(y - x^2) = 0$ so

+ Sometimes polynomial functions do not look polynomial

• $V = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2_{\mathbb{R}}$

$f : V \longrightarrow \mathbb{R}$
$(x,y) \longmapsto \frac{1}{x}$

$f = F|_V \qquad F : \mathbb{A}^2 \longrightarrow \mathbb{R}$
$(x,y) \longmapsto y$

• $V = x^2 + y^2 - 1 \subseteq \mathbb{A}^2_{\mathbb{R}}$

$g : V \longrightarrow \mathbb{R}$
$(x,y) \longmapsto e^{x^2+y^2}$

$g = G|_V \qquad G : \mathbb{A}^2 \longrightarrow \mathbb{R}$
$(x,y) \longmapsto e$

• **Lemma**: Let $K \subseteq \mathbb{C}$. An algebraic subset $V \subseteq \mathbb{A}^n_K$ is closed in the Euclidean topology on K^n.

Pf: $V = \mathbb{V}(f_1, \dots, f_t) = \bigcap_{i=1}^{t} \mathbb{V}(f_i)$
HBT

$$= \bigcap_{i=1}^{t} f_i^{-1}(\{0\})$$

closed in Euclidean top

continuous in Euclidean top.

• **Def**: The Zariski-topology on $\mathbb{A}^n_K$ is given by letting the closed sets be the algebraic sets.

• **Lemma**: This is a topology because

1) $\mathbb{V}(\langle 0 \rangle) = \mathbb{A}^n_K$
2) $\mathbb{V}(\langle 1 \rangle) = \emptyset$
3) $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(IJ)$
4) $\mathbb{V}(\{F_i\}_{i \in \Delta}) \cup \mathbb{V}(\{G_i\}_{j \in \Delta'}) = \mathbb{V}(\{F_i, G_i\}_{i \in \Delta \cup \Delta'})$

• The Zariski topology on an algebraic set $V \subseteq \mathbb{A}^n_K$ is the subspace topology; closed sets are of the form

$$V \cap Z$$

for $Z \subseteq \mathbb{A}^n_K$ an algebraic set.

• **Fact**: $S \subseteq \mathbb{A}^n_K$ then $\bar{S} = \mathbb{V}(\mathbb{I}(S))$.

- Zariski closed $\Rightarrow$ Euclidean closed
- Zariski open $\Rightarrow$ Euclidean open

} converse is false

- **Def**: Let $V \subseteq \mathbb{A}^n_k$ and $W \subseteq \mathbb{A}^m_k$. A morphism of algebraic varieties from V to W is a function $V \xrightarrow{f} V$ such that there exists polynomial map

$$\begin{tikzcd} \mathbb{A}^n \arrow[r, "F"] & \mathbb{A}^m \\ V \arrow[r, "f"] \arrow[u, hook] & W \arrow[u, hook] \end{tikzcd}$$

- Here by polynomial map we mean $(x_1, \dots x_n) \longmapsto (F_1(\bar{x}), F_2(\bar{x}), \dots, F_m(\bar{x}))$ for polynomials $F_i \in \mathbb{C}[x_1, \dots x_n]$.

- **Remarks** +) morphisms are cts in the Zariski topology

  +) morphisms need not be closed, i.e. send subvarieties to subvarieties

$$\mathbb{V}(xy-1) \subseteq \mathbb{A}^2 \longrightarrow \mathbb{A}^1 \supsetneq \mathbb{A}^1 \setminus \{0\}.$$
$$(x,y) \longmapsto y$$

- **Def**: Two algebraic sets V and W are isomorphic $\Leftrightarrow$ there exists morphisms

$$V \xrightarrow{f} W \qquad W \xrightarrow{g} V$$

such that $g \circ f = Id_V$ and $f \circ g = Id_W$.

• A f.g. reduced $\mathbb{k}$-algebra $\cong \mathbb{k}[x_1, \dots, x_n]/I$

for $I \subseteq \mathbb{k}[x_1, \dots, x_n]$ radical ideal.

• Thm: (Nullstellensatz II): Assume $\mathbb{k} = \overline{\mathbb{F}}$

1) If $F: V \longrightarrow W$ is a morphism then

$$\begin{array}{ccc} \mathbb{C}[W] & \xrightarrow{F^*} & \mathbb{C}[V] \\ g & \longmapsto & g \circ F \end{array}$$

is a morphism of $\mathbb{k}$-algebra.

2) If $\mathbb{C}[y_1, \dots, y_m]/J \xrightarrow{\sigma} \mathbb{C}[x_1, \dots, x_n]/I$ ~~is~~ a morphism of $\mathbb{k}$-algebras w/ J and I radical, let $F_i \in \mathbb{C}[x_1, \dots, x_n]$ be any polynomial such that $\sigma(y_i) = F_i \bmod I$. Then

$$\mathbb{A}^n_{\mathbb{k}} \xrightarrow[(F_1 \dots F_m)]{F} \mathbb{A}^m_{\mathbb{k}}$$

is a polynomial morphism that sends $\mathbb{V}(I)$ to $\mathbb{V}(J)$.

This induced map is independent of choice

3) The map F from (2) has $F^* = \sigma$.

4) ~~V~~ and W are isomorphic $\Longleftrightarrow$ $\mathbb{C}[V] \cong \mathbb{C}[W]$ as $\mathbb{k}$-algebras

5) There is an anti-equivalence of categories

$$\left\{ \begin{array}{c} \text{algebraic sets} \\ \text{morphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{f.g. reduced} \\ \mathbb{k}\text{-algebras} \end{array} \right\}$$

- **Def**: Projective n-space over $\mathbb{R}$ is the set

$$\mathbb{P}^n_{\mathbb{R}} = \{\text{one-dimensional subspaces } \mathbb{R}^{n+1}\}$$

- **Lemma**:

$$\mathbb{P}^n_{\mathbb{R}} = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$$

where $(a_0, \dots a_n) \sim (b_0, \dots b_n)$

$\iff \exists \lambda \in \mathbb{R}^\times \quad (\lambda a_0, \dots \lambda a_n) = (b_0, \dots b_n)$

- A point in $\mathbb{P}^n_{\mathbb{R}}$ can be thought of as an equivalence classe

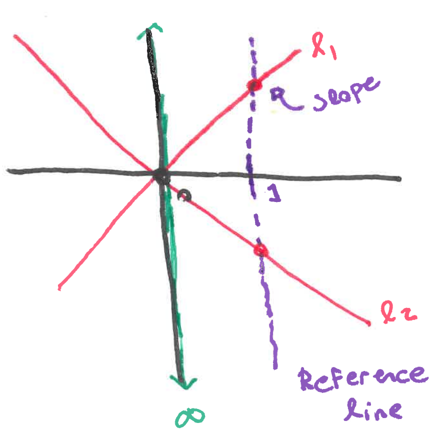
$$[(a_0, \dots, a_n)] = \{(\lambda a_0, \dots \lambda a_n) \mid \lambda \in \mathbb{C}^*\}$$

with some $a_i \neq 0$. We write $[a_0 : a_1 : \dots a_n]$ for this class and call $a_0, \dots a_n$ the homogeneous coordinates.

only defined up to scaling by non-zero #!.

- **Ex**: $[1:2:3] = [6:12:18] \neq [5:12:18]$.

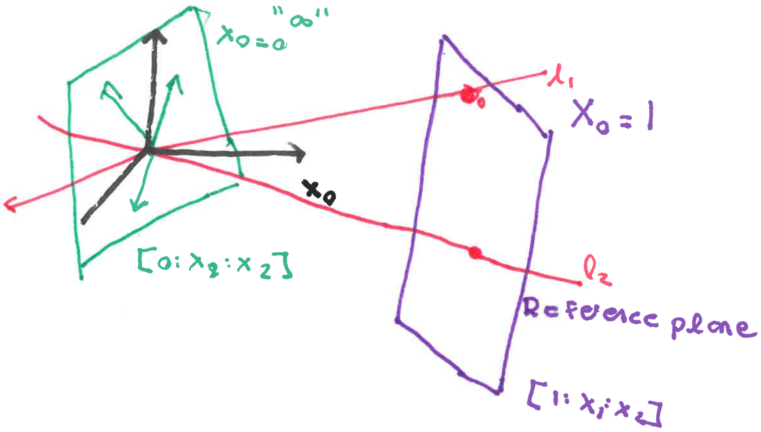
- **Ex**: $\mathbb{P}^1_{\mathbb{C}}$



$$\mathbb{P}^1 \longrightarrow \mathbb{C} \cup \{\infty\}$$

$$[x_0 : x_1] \longmapsto \begin{cases} \frac{x_1}{x_0} & x_0 \neq 0 \\ \infty & x_0 = 0 \end{cases}$$

• **Ex**: $\mathbb{P}^2_{\mathbb{C}}$



$$\mathbb{P}^2 \longrightarrow \mathbb{P}^1 \cup \mathbb{C}^2.$$

$$[x_0:x_1:x_2] \longmapsto \begin{cases} \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) & x_0 \neq 0 \\ [x_1:x_2] & x_0 = 0 \end{cases}$$

$$\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{C}^1 \cup \{\infty\}$$

• The above idea generalizes.

• **Lemma**: For $i = 0, 1, \ldots n$ let

$$U_i = \{[x_0 : \cdots x_n] \in \mathbb{P}^n_{\mathbb{R}} \mid x_i \neq 0\}$$

1) There is a bijection

$$\mathbb{P}^n_{\mathbb{R}} \xrightarrow{\ \tau_i\ } U_i \cup \mathbb{P}^{n-1}_{\mathbb{R}}$$

$$[x_0 : \cdots x_n] \longmapsto \begin{cases} \left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right) & x_i \neq 0 \\ [x_0 : \cdots \widehat{x_i} : \cdots x_n] & x_i = 0 \end{cases}$$

2) $\tau|_{U_i}$ is a bijection w/ U_i

3) $\mathbb{P}^n_{\mathbb{R}} = \bigcup_{i=0}^{\infty} U_i$

4) $\tau_j \circ \tau_i^{-1} : \tau_i(U_i \cap U_j) \longrightarrow \tau_j(U_i \cap U_j)$ are rational functions in the cordinates on $\mathbb{A}^n_{\mathbb{R}}$.

• Ex: • $\psi_n \circ \psi_0^{-1}(a_1, \dots a_n)$

$$= \psi_n(\psi_0^{-1}(a_1, \dots a_n)) = \psi_n([1, a_1 : \dots a_n])$$

$$= \left(\frac{1}{a_n}, \frac{a_1}{a_n}, \dots \frac{a_{n-1}}{a_n}\right)$$

• $\psi_j \circ \psi_i^{-1}(a_1, \dots a_n)$

$$= \psi_j(\psi_i^{-1}(a_1, \dots a_n)) = \psi_j([a_1, \dots a_{i-1}, 1, a_i, \dots a_n])$$

$$= \left(\frac{a_1}{a_j}, \dots \frac{a_{i-1}}{a_j}, \frac{1}{a_j}, \dots \frac{\hat{a}_j}{a_j}, \dots \frac{a_n}{a_j}\right)$$

• Fact: $R = \mathbb{C}/\mathbb{R}$, makes $\mathbb{P}^n_R$ a compact $\mathbb{R}/\mathbb{C}$-manifold.

• Fact: There no non-constant functions $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{C}$

• Def: A polynomial $F \in \mathbb{C}[x_0, \dots x_n]$ is homogeneous

$\iff$ every term has same degree

$\iff$ $F(\lambda x_0, \dots \lambda x_n) = \lambda^{\deg(F)} F(x_0, \dots x_n) \quad \forall \lambda \in k^*$

• Ex: Warning homogeneous polynomials are not functions on $\mathbb{P}^n$

$$\mathbb{P}^1 \longrightarrow \mathbb{C}$$
$$[x_0 : x_1] \longmapsto x_0$$
$$[1:1] = [2:2] \longmapsto 1 \neq 2 \quad \lightning$$

- However, given $f_1, \dots f_t \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous

$$\mathbb{V}(f_1, \dots . f_t) = \{ a \in \mathbb{P}^n_k \mid f_1(\bar{a}) = \cdots = f_t(\bar{a}) = 0 \}$$

is well-defined subset of $\mathbb{P}^n_k$.

- <u>Def</u>: A set of the form $\mathbb{V}(f_1 \dots f_t)$ is a <u>projective variety</u>.

  More generally of the form

$$\mathbb{V}(I) = \{ a \in \mathbb{P}^n_k \mid f(\bar{a}) = 0 \quad \forall f \in I \}$$

  for a <u>homogeneous ideal</u> $I \subseteq \mathbb{C}[x_0, \dots x_n]$

  $\hookrightarrow$ ideal generated by homogeneous elements.

- <u>Def</u>: If $V \subseteq \mathbb{P}^n_k$ is a projective variety.

$$\mathbb{I}(V) = \{ f \in \mathbb{C}[x_0, \dots x_n] \mid f(a) = 0 \quad \forall a \in V \}$$

- <u>Lemma</u>: $\mathbb{I}(V)$ is an ideal.

- <u>Thm</u>: (Projective Nullstellensatz): Assume $k = \bar{k}$

  1) Let $I \subseteq k[x_0, \dots x_n]$ then

$$\mathbb{V}(I) = \emptyset \iff \langle x_0, \dots x_n \rangle^t \subseteq I \quad \text{for some } t \geq 0$$

  2) If $\mathbb{V}(I) \neq \emptyset$ then

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$$

Thm (cont.)

3) There is a inclusion reversing bijection

$$\left\{ \begin{matrix} \text{Projective subvarieties} \\ \mathbb{P}^n_k \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{radical homogeneous ideals} \\ \neq \langle x_0, \dots, x_n \rangle \text{ in} \\ k[x_0, \dots, x_n] \end{matrix} \right\}$$

- Facts 1) If $k \subseteq \mathbb{C}$ then projective varieties are compact in the Euclidean topology

2) Projective varieties are the compactifications (in both topologies) of affine varieties

3) Projective subvarieties are covered by ~~alg~~ algebraic sets

- Def : The homogeneous coordinate ring of $V \subseteq \mathbb{P}^n_k$ is $\mathbb{C}[x_0, \dots x_n]/\mathbb{I}(V)$

warning these are not functions on V.

- Def : The Zariski topology on $\mathbb{P}^n_k$ is given by letting the closed sets be projective subvarieties. The Zariski topology on $V \subseteq \mathbb{P}^n_k$ is the subspace topology.

- Def : A morphism between projective subvarieties $V \subseteq \mathbb{P}^n$, $W \subseteq \mathbb{P}^m$ is a function $V \xrightarrow{F} W$ if for all $p \in V$ there exists $F_0, \dots F_m \in \mathbb{C}[x_0, \dots x_n]$ homogeneous of the same degree and an Zariski open neighborhood $p \in U \subseteq V$ s.t.

$$F|_U = \left[\begin{matrix} U \longrightarrow \mathbb{P}^m \\ \bar{x} \longmapsto [F_0(\bar{x}) : \dots : F_m(\bar{x})] \end{matrix} \right]$$

- **Ex**: $C = \mathbb{V}(zx - y^2) \subseteq \mathbb{P}^2$

$$\begin{aligned} C &\longrightarrow \mathbb{P}^1 \\ [x:y:z] &\longmapsto \begin{cases} [x:y] & x \neq 0 \\ [y:z] & z \neq 0 \end{cases} \end{aligned}$$

If $[x:y:z] \in C$ then $x=0$ or $z=0$ but not both

Further if $x \neq 0$ and $z \neq 0$

$$[x:y] = [xy : y^2] = [xy : xz] = [y:z]$$

- **Def**: Two projective subvarieties are <u>isomorphic</u> iff there exists mutual inverse morphisms

- **Ex**: $\mathbb{P}^1 \longrightarrow C$

$[s:t] \longmapsto [s^2 : st : t^2]$ verify this is the inverse to the above

- **Ex**: Warning C and $\mathbb{P}^1$ are isomorphic but

$$\frac{\mathbb{C}[x,y,z]}{\langle xz - y^2 \rangle} \not\cong \mathbb{C}[s,t]$$

• **Recall**: A locally closed subset of a top. space X is a closed subset of an open subset, i.e.

$$\underset{\text{open}}{U} \cap \underset{\text{closed}}{Z} \subseteq X$$

• **Def**: A quasi-projective variety is a locally closed subset $\mathbb{P}^n_k$.

• ~~Def~~

Lemma: The following are quasiprojective varieties:

1) Projective varieties
2) ~~affine varieties~~ algebraic sets
3) open subsets of the above.

• **Ex**: 1) $\mathbb{A}^1_k \setminus \{0\} \subseteq \mathbb{A}^1_k \subseteq \mathbb{P}^1$ is quasiprojective.

2) $(k^*)^n = \mathbb{P}^n \setminus \mathbb{V}(x_0 x_1 \cdots x_n)$ is quasiprojective.

• **Pf**: If $f \in k[x_1, \dots x_n]$ then $f^h = x_0^d f\left(\frac{x_1}{x_0}, \dots \frac{x_n}{x_0}\right) \in k[x_0, \dots x_n]$ where $d = \deg(f)$. Note f^h is homg of degree d

• If $x_0 \neq 0$ then $f^h(x_0, \dots x_n) = 0 \iff f\left(\frac{x_1}{x_0}, \dots \frac{x_n}{x_0}\right) = 0$

So if $V = \mathbb{V}(f_1, \dots f_t) \subseteq \mathbb{A}^n_k$ then

$$V = \mathbb{V}(f_1^h, \dots, f_t^h) \cap \underbrace{D(x_0)}_{x_0 \neq 0} \subseteq \mathbb{P}^n_k$$

- **Def**: A morphism of quasi-projective varieties $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ if there is a function $F : V \longrightarrow W$ st $\forall p \in V$ there exists homogeneous polynomials $F_0, \dots, F_m \in k[x_0, \dots, x_n]$ of the same degree and a open subset $p \in U \subseteq V$ such that

$$F|_U = \begin{array}{rcl} V & \longrightarrow & W \\ p & \longmapsto & [F_0(p) : \cdots : F_m(p)]. \end{array}$$

- **Ex**: $V = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2$ and $W \subseteq \mathbb{A}^1 \setminus \{0\}$

$$\begin{array}{rcl} W & \xrightarrow{F} & V \\ t & \longmapsto & (t, \frac{1}{t}) \end{array}$$

view $W \subseteq \mathbb{P}^1$ by $t \mapsto [t:1]$ and $V = \mathbb{V}(xy - z^2) \subseteq \mathbb{P}^2$
$z \neq 0$
$(x,y) \longmapsto [x:y:1]$

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\tilde{F}"] & \mathbb{P}^2 \\ {[a:b]} \arrow[r, mapsto] & {[a^2 : b^2 : ab]} \\ W \arrow[u, hook] \arrow[r, "F"] & V \arrow[u, hook] \end{tikzcd}$$

$t = \frac{a}{b}$ $\quad \tilde{F}([t:1]) = [t^2 : 1 : t]$
$= [t : \frac{1}{t} : 1]$

- **Def**: A quasi-projective variety is affine iff it is isomorphic to some algebraic set $V \subseteq \mathbb{A}^n_k$ for some n.

- **Def**: If W is a quasi-projective affine variety then the coordinate ring of W is $\mathbb{C}[W] \cong \mathbb{C}[V]$ for any $V \subseteq \mathbb{A}^n_k$ w/ $W \cong V$.

- <u>Warning</u>: We must check that if $V \subseteq \mathbb{A}^n_k$ and $W \subseteq \mathbb{A}^n_k$ are algebraic sets isomorphic as quasiprojective varieties $\Rightarrow$ V & W are isomorphic as algebraic set

- <u>Ex</u>: $\mathbb{C}[\mathbb{A}^1 \setminus 0] \cong \dfrac{\mathbb{C}[x,y]}{\langle xy-1\rangle} \cong \mathbb{C}[t, t^{-1}]$

- <u>Fact</u>: We could redefine projective variety, but this does not change things.

- <u>Ex</u>: Let $V = \mathbb{V}(g_1, \dots g_t) \subseteq \mathbb{A}^n_k$ and $f \in \mathbb{C}[V]$ then the set $U_f = V \setminus \mathbb{V}(f)$ is quasiprojective.

- <u>Lemma</u>: If $V = \mathbb{V}(g_1, \dots g_t) \subseteq \mathbb{A}^n_k$ and $f \in \mathbb{C}[V]$ then $U_f = V \setminus \mathbb{V}(f)$ is affine and

$$\mathbb{C}[U_f] \cong \mathbb{C}[V][\tfrac{1}{f}]$$

- <u>Pf</u>: Think of $U_f \subseteq V \subseteq \mathbb{A}^n$ and consider the map

$$\begin{aligned} U_f &\xrightarrow{F} \mathbb{A}^{n+1}_{x_1, \dots x_n, z} \\ (x_1, \dots x_n) &\longmapsto \left(x_1, \dots x_n, \frac{1}{f(x_1, \dots x_n)}\right) \end{aligned}$$

This is well-defined since $f \neq 0$ on U_f. The image is

$$\mathbb{V}(g_1, \dots g_t, fz-1)$$

Argue as above

$\blacksquare$

- Warning: $\mathbb{A}^2 \setminus \{0\} = \mathbb{A}^2 \setminus \langle x,y \rangle$ is not affine

- Lemma: Every quasiprojective variety is covered by affine varieties.

  Pf: View $V \subseteq \mathbb{P}^n_k$ then $V \cap U_i \subseteq U_i \cong \mathbb{A}^n$ is quasiprojective

  and then $V \cap U_i = \mathbb{V}(f_1, \dots f_t) \setminus \mathbb{V}(g_1, \dots g_n)$ for $f_i, g_j \in k[x_1, \dots x_n]$

  these are then covered by $\mathbb{V}(f_1, \dots f_t) \setminus \mathbb{V}(g_i)$

- Def: Let $U \subseteq V$ be a open subset of an affine variety k, and $U \xrightarrow{f} k$ a function
  1) the function is regular at $p \in U$, iff $\exists\, h, g \in \mathbb{C}[V]$ such that $h(p) \neq 0$ and $\exists\, U' \ni p$ open st. $f|_{U'} = \frac{g}{h}$.
  2) f is regular on U iff it is regular $\forall p \in U$.

  write $\mathcal{O}_V(U)$ for the ring of regular functions wrt pointwise addition/multiplication

- Ex: $U = \mathbb{A}^2 \setminus \mathbb{V}(x) \longleftrightarrow k$

  $(x,y) \longmapsto \frac{y}{x}$

- Thm: Let $V \subseteq \mathbb{A}^n_k$ be an alg. set then $\mathbb{C}[V] \cong \mathcal{O}_V(V)$.

  global regular functions are restrictions of a global function $\mathbb{A}^n \to k$

- Def: Let $U \subseteq V$ be an open subset of a quasiprojective variety, and $U \xrightarrow{f} k$ a function
  1) the function is regular at $p \in U$ iff $\exists\, p \in U' \subseteq U$ w/ U' affine st. f is regular at $p \in U'$
  2) f is regular on U iff it is regular $\forall p \in U$

  write $\mathcal{O}_V(U)$ for the ring of regular functions on U.

- **Lemma**: 1) $\mathcal{O}_V$ is a sheaf of k-algebras

  2) If $F: V \xrightarrow{F} W$ is a morphism of quasiprojective varieties for all $U \subseteq W$ open and $f \in \mathcal{O}_W(U)$

$$f \circ F \in \mathcal{O}_V(F^{-1}(U))$$

i.e. F induces a map of sheaves.

- **Def**: A function $V \xrightarrow{\phi} W$ of quasi-projective varieties is a morphism $\Leftrightarrow$ $\forall p \in V$ there exists affine open neighborhoods $p \in U$ and $\phi(p) \in U'$

$$\begin{tikzcd} V \arrow[r, "\phi"] & W \\ U \arrow[u, hook] \arrow[r] & U' \arrow[u, hook] \\ p \arrow[u, phantom, "\in" rotate=90] & \phi(p) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

map of affines i.e. given by regular functions

- **Def**: An abstract algebraic variety is a top. space V and a sheaf of k algebras $\mathcal{O}_V$ s.t. there is an open cover $V = \bigcup U_i$ where $(U_i, \mathcal{O}_V|_{U_i})$ is isomorphic to $(W_i, \mathcal{O}_{W_i})$ w/ this affine

- Informally $\Big(\underbrace{\{V_\alpha\}}_{\text{affine variety}}, \underbrace{\{V_{\alpha\beta}\}}_{\text{affine open subset}}, \underbrace{\{g_{\alpha\beta}\}}_{\text{isomorphisms } g_{\alpha\beta}: V_{\alpha\beta} \longrightarrow V_{\beta\alpha}}\Big)$

$+ \; g_{\alpha\alpha} = 1$

$+ \; g_{\beta\gamma}\big|_{V_{\beta\alpha} \cap V_{\beta\gamma}} \circ g_{\alpha\beta}\big|_{V_{\alpha\beta} \cap V_{\alpha\gamma}} = g_{\alpha\gamma}\big|_{V_{\alpha\beta} \cap V_{\alpha\gamma}}$

$X = \bigcup V_\alpha / \sim \qquad a \sim b \Leftrightarrow g_{\alpha\beta}(a) = b$