DAILY UPDATE: MATH 121: COMMUTATIVE ALGEBRA (SPRING 2026)

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Contents

1. Week 1

1.1. **Tuesday, March 31, 2026.** After going over the syllabus we reviewed the basics of ring theory: rings, ideals, ring homomorphisms, units, zerodivisors, kernels, and images, etc. We focussed on four main examples: i) the ring of integers $\mathbb{Z}$, ii) the polynomial ring $\mathbb{C}[x_0,\dots,x_n]$, iii) the ring of formal power series $R[[t]]$, and 4) the ring of functions on a "geometric" space X $\operatorname{Hom}(X,\mathbb{R})$. Please read Chapter 0 of Eisenbud or Reid, and Section 1.1 of Hochster.

1.2. **Thursday, April 2, 2026.** We reviewed the fundamental theorem of quotient rings, which proved that all ideals arise as the kernel of some ring homomorphism, ideals in R/I are in bijection with ideals in R containing I, and $\mathrm{Hom}(R/I.S)$ is in bijection with homomorphisms $R \to S$ such that $I \subset \ker(\phi)$. We discussed why this last property is called the universal property of quotient rings, as it uniquely characterizes them up to isomorphism. We then discussed prime and maximal ideals, and introduced $\mathrm{Spec}(R)$ and $\mathrm{mSpec}(R)$, the set of prime ideals in R and maximal ideals in R. The next few weeks will focus on studying these sets. For example, we ended part way through the proof that $\mathrm{mSpec}(R)$ and $\mathrm{Spec}(R)$ are non-empty. Please ready §1.2-1.9 of Reid, Chapter 1 of Eisenbud, and §1.4 of Hochster.

1.3. **Friday, April 3, 2026.** We finished the proof that every non-zero ring has a maximal ideal. The proof uses Zorn's lemma applied to the set of proper ideals with respect to inclusion. The difficult part of the proof was verifying each chain has an upper bound, which required the previously stated fact that $I \cup J$ is an ideal if and only if $I \subset J$ or $J \subset I$. We the discussed how every proper ideal $I \subset R$ is contained in a prime ideal. The ideal was to use that R/I has a maximal ideal $P \subset R/I$ and if $\pi : R \to R/I$ is the quotient then $\pi^{-1}(P)$ is a proper ideal containing I. The pre-image of a maximal ideal need not be maximal, but the preimage of a prime ideal is prime. We then worked on our first worksheet exploring Gröbner bases, where we explored the division algorithm in $\mathbb{K}[x]$ and how to implement it in *Macaulay2*. In order to generalize this to a division algorithm for $\mathbb{K}[x_1,\ldots,x_n]$ we will need to fix a *monomial order*, which we began exploring. We stopped at question 6 on the worksheet and will finish the remainder later. Optional: If you would like to learn more about monomial orders and the division algorithm read some of Chapter 15 of Eisenbud or Chapter 1 of Cox, Little, O'Shea.

2. Week 2

2.1. **Tuesday, April 7, 2026.** We began exploring both $\mathrm{Spec}(R)$ and $\mathrm{mSpec}(R)$ as topological spaces, by introducing the *Zariski topology*. Given a subset $\mathscr{S} \subset R$ we define $\mathbb{V}(\mathscr{S})$ to be the set of prime ideals in R containing $\mathscr{S}$. The Zariski topology on $\mathrm{Spec}(R)$ is defined by declaring these to be the closed sets. The Zariski topology on $\mathrm{mSpec}(R)$ is the subspace topology from $\mathrm{mSpec}(R) \subset \mathrm{Spec}(R)$. We explored some examples of this topology, $\mathrm{Spec}(\mathbb{Z})$, $\mathrm{Spec}(\mathbb{K}[x])$ with $\mathbb{K}$ algebraically closed, and then study certain special points called dense points. We then showed that if $\phi : R \to S$ is a ring map there is an induced map $\phi^{\#} : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ given by $[P] \mapsto [\phi^{-1}(P)]$. Be sure you check that $\phi^{\#}$ is continuous in the Zariski topology. In summary, we see that Spec is a functor from the category of rings to the category of topological spaces. **Expect a short quiz on Thursday on primes and $\mathrm{Spec}(\mathbb{Z})$.**

2.2. **Thursday, April 9, 2026.** Today in class we took our first quiz, which was about computing vanishing sets in $\mathrm{Spec}(\mathbb{Z})$ and $\mathrm{Spec}(\mathbb{K}[x])$. We then discussed $\mathbb{K}$-algebras, which is just a ring R together with a fix ring homomorphism $\alpha : \mathbb{K} \to R$. We then shifted to looking at algebraic ssets, which are just subsets of $\mathbb{K}^n$ defined by the vanishing of a set $\mathbb{V}(\mathscr{S})$ of a set of polynomials $\mathscr{S} \subset \mathbb{K}[x_1,\dots,x_n]$. Warning: We are overloading the $\mathbb{V}$ notation as we now have to different things are calling $\mathbb{V}(-)$. We also defined the defining ideal $\mathbb{I}(V) \subset \mathbb{K}[x_1,\dots,x_n]$ for an algebraic set $V \subset \mathbb{K}^n$. Concretely:

$$\mathbb{V}(\mathscr{S}) = \{p \in \mathbb{K}^n \mid f(p) = 0 \text{ for all } f \in \mathscr{S}\} \quad \text{and} \quad \mathbb{I}(V) = \{f \in \mathbb{K}[x_1,\dots,x_n] \mid f(p) = 0 \text{ for all } p \in V\}.$$

We verified some basic properties about these operators, and then discussed Hilbert's Nullstellensatz, which says among many other things, that over an algebraically closed fields these operations are mutual inverse when restricted to the set of radical ideals. We then discussed the Zariski topology on algebraic sets, before ending with the coordinate ring of an algebraic set. The coordinate ring $\mathbb{K}[V]$ of an algebraic set is the ring of functions $V \to \mathbb{K}$ that are the restriction of a global polynomial function.

3. Week 3

3.1. **Tuesday, April 14, 2026.** We returned to Gröbner bases. The main goal was to develop the multivariable division algorithm and use it to prove that every ideal of $\mathbb{K}[x_1,\dots,x_n]$ is finitely generated. We stated and worked through the multivariable division algorithm, which given a polynomial f and an ordered list $G = (g_1,\dots,g_s)$, produces quotients $q_1,\dots,q_s$ and a remainder r satisfying $f = q_1g_1 + \cdots + q_sg_s + r$, where no monomial in r is divisible by any $\mathrm{LM}(g_i)$. Working through explicit examples revealed a key subtlety: unlike the one-variable case, the remainder depends on the ordering of the divisors in G, and a nonzero remainder does *not* certify that $f \notin \langle g_1,\dots,g_s\rangle$. To address this, we studied monomial ideals, where membership is controlled purely by divisibility. We stated and proved Dickson's Lemma — that every subset of $\mathbb{Z}^n_{\geq 0}$ has finitely many minimal elements — by induction on n, and used it to show every monomial ideal is finitely generated. We then defined the initial ideal $\mathrm{in}_<(I) = \langle \mathrm{LT}(f) \mid f \in I\rangle$ and Gröbner bases: a finite subset $\mathscr{G} \subset I$ such that the leading terms of the elements of $\mathscr{G}$ generate $\mathrm{in}_<(I)$. Using Dickson's Lemma we proved that Gröbner bases exist, and showed that dividing by a Gröbner basis *does* solve the ideal membership problem: $f \in I$ if and only if the remainder on dividing f by $\mathscr{G}$ is zero. This also showed every Gröbner basis generates I, proving that every ideal of $\mathbb{K}[x_1,\dots,x_n]$ is finitely generated. We then defined Noetherian rings via the ascending chain condition, showed this is equivalent to every ideal being finitely generated, and stated Hilbert's Basis Theorem: if R is Noetherian then so is $R[x]$. We worked on the proof, which uses an ascending chain argument on leading coefficients that closely parallels the proof of Dickson's Lemma.

3.2. **Thursday, April 16, 2026.** Salim Tayou was our guest lecturer today, and class focused on proving Hilbert's Nullstellensatz under the extra hypothesis the field is uncountable. The content of the Nullstellensatz is the inclusion $\mathbb{I}(\mathbb{V}(I)) \subseteq \sqrt{I}$ when $\mathbb{K}$ is algebraically closed. We first proved Hilbert's Nullstellensatz I — that every maximal ideal of $\mathbb{K}[x_1,\dots,x_n]$ is of the form $\langle x_1 - a_1,\dots,x_n - a_n\rangle$ — under the additional assumption that $\mathbb{K}$ is uncountable. The argument used field theory: if $\mathfrak{m}$ is maximal then $L = R/\mathfrak{m}$ is a field extension of $\mathbb{K}$, and since R has countably many monomials, $\dim_{\mathbb{K}} L$ is at most countable. A clever linear independence argument shows that any transcendental element of L would force $\dim_{\mathbb{K}} L$ to be uncountable, so the extension must be algebraic, hence trivial when $\mathbb{K}$ is algebraically closed. From this we deduced the weak Nullstellensatz: $\mathbb{V}(I) = \varnothing$ if and only if $I = \langle 1 \rangle$. We then used Rabinowitsch's trick to deduce the full Nullstellensatz II from the weak version. Lastly, we discussed the relative Nullstellensatz III connecting algebraic subsets of an algebraic set V with radical ideals of its coordinate ring $\mathbb{K}[V]$, and the correspondence between irreducible algebraic sets and prime ideals.

4. Week 4

4.1. **Tuesday, April 21, 2026.** Today we began the study of modules. An *R-module* is an abelian group $(M, +)$ together with a scalar multiplication $R \times M \to M$ satisfying associativity, distributivity (over both R and M), and a unital identity axiom. In other words, a module is a vector space where the scalars come from a ring instead of a field. We discussed many basic examples: i) if $\mathbb{K}$ is a field then a $\mathbb{K}$-module is the same thing as a $\mathbb{K}$-vector space, ii) every ring R is a module over itself via its own multiplication, iii) if $I \subset R$ is and ideal then both I and R/I are R-modules, and iv) a $\mathbb{Z}$-module is the same thing as an abelian group. Thus, modules simultaneously generalizes both linear algebra and abelian groups. We then introduced module homomorphisms, which are group homomorphisms $\phi : M \to N$ that respect the R-action, i.e. $\phi(rm) = r\phi(m)$. As in the ring case, kernels and images are defined in the natural way, and we stated the First and Third Isomorphism Theorems for modules, which required introducing the quotient module M/N for a submodule $N \subseteq M$, The worksheet also covers direct sums and products, free modules, Hom as a module, and tensor products. **Expect a quiz on Thursday.**

4.2. **Thursday, April 23, 2026.** Today we took a quiz on multivariable polynomial division, please be sure this makes sense. After that we introduced *localization*, the process of forcing a multiplicatively closed set $U \subset R$ to become invertible. We defined $U^{-1}R$ using formal fractions r/u, discussed the equivalence relation needed when zero divisors are present, and proved basic properties such as when the localization map $R \to U^{-1}R$ is injective. We emphasized the universal property of localization and used it to understand the main examples: fields of fractions, localizations $R_{\mathfrak{p}}$ at prime ideals, and rings of the form $R[\frac{1}{f}]$, including the identification $R[\frac{1}{f}] \cong R[x]/\langle fx - 1 \rangle$. We then studied extension and contraction of ideals under a ring map, observing that they are not inverse operations in general, but become very well-behaved for prime ideals after localization: prime ideals of $U^{-1}R$ correspond exactly to prime ideals of R disjoint from U. In particular, the primes of $R_{\mathfrak{p}}$ correspond to primes contained in $\mathfrak{p}$, while the primes of $R[\frac{1}{f}]$ correspond to primes not containing f. We ended by extending localization from rings to modules, constructing $U^{-1}M$, discussing its universal property, and identifying it with $U^{-1}R \otimes_R M$. **Next week there will be a take-home quiz you submit via Canvas.**

4.3. **Friday, April 24, 2026.** Today we returned to the Zariski topology using localization as our main tool. We introduced the basic open sets $D(f) = \{[P] \in \operatorname{Spec}(R) \mid f \notin P\}$, proved that they form a basis for the Zariski topology, and studied the map $\eta^\# : \mathrm{Spec}(U^{-1}R) \to \mathrm{Spec}(R)$ induced by localization. We showed that this map is a homeomorphism onto its image $X_U = \{[\mathfrak{p}] \in \mathrm{Spec}(R) \mid \mathfrak{p} \cap U = \emptyset\}$, and in the special case $U = \{1, f, f^2, \ldots\}$ this identifies $\mathrm{Spec}(R[\frac{1}{f}])$ with the basic open $D(f)$. We then used basic opens to prove that $\mathrm{Spec}(R)$ is quasi-compact, translated basic-open covers into unit-ideal statements, and discussed how closed subsets are always quasi-compact while open subsets can fail to be quasi-compact in non-Noetherian examples. Next we revisited nilpotents, proving the characterization $\mathrm{Nil}(R) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$ and showing that $\mathrm{Spec}(R_{\mathrm{red}}) \to \mathrm{Spec}(R)$ is a homeomorphism, capturing the motto that geometry does not see nilpotents. Finally, we introduced residue fields $\kappa(\mathfrak{p})$ and the fibre ring $\kappa(\mathfrak{p}) \otimes_R S$, which describes the fibre of a map $\phi^\# : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ over the point $[\mathfrak{p}]$. **Your take-home quiz will be posted to Canvas tonight, it will be due next Friday, May 1st via Canvas.**

5. Week 5

5.1. **Tuesday, April 28, 2026.** This week is a review and settling week. The goal is to give us time to digest and master our main new tools. Today we reviewed some exercises with tensor products and homs. **Your take-home quiz is posted to Canvas, it is due Friday, May 1st via Canvas.**

5.2. **Thursday, April 30, 2026.** This week is a review and settling week. The goal is to give us time to digest and master our main new tools. Today we reviewed some exercises with localization, tensor products and homs. **Your take-home quiz is posted to Canvas, it is due Friday, May 1st via Canvas.**

5.3. **Friday, May 1, 2026.** No class today because of AGNES. **Your take-home quiz is posted to Canvas, it is due tonight, Friday , May 1st via Canvas.**

6. Week 6

6.1. **Tuesday, May 5, 2026.** Today we introduced the language of exactness for sequences of R-modules. We defined complexes, exactness at a module, homology $H_i = \ker(\phi_i)/\mathrm{img}(\phi_{i+1})$, and short exact sequences $0 \to A \to B \to C \to 0$, emphasizing that they identify A with a submodule of B and C with the corresponding quotient. We recast familiar constructions such as ideals, quotient modules, multiplication by n, kernels, images, and cokernels in this language, and introduced presentations of modules by free modules. We then studied split short exact sequences, proved the Splitting Lemma, and saw examples where splitting always occurs for vector spaces but can fail for $\mathbb{Z}$-modules. Finally, we organized $\mathrm{Hom}_R(T, -)$, $\mathrm{Hom}_R(-, T)$, tensor product, and localization as functors. We proved that the two Hom functors are left exact, tensor product is right exact but not necessarily left exact, defined flat modules as those for which tensoring is exact, and used a denominator-clearing argument to prove that localization preserves exact sequences and therefore that $U^{-1}R$ is a flat R-module.

6.2. **Thursday, May 7, 2026.** Today we continued studying exactness with localization, proving that an R-module is zero if and only if its localizations at all maximal ideals, equivalently all prime ideals, vanish, and that homology commutes with localization. This gave the slogan that exactness can be checked locally: kernels, cokernels, injectivity, surjectivity, and isomorphisms can be tested after localizing at all maximal ideals, all prime ideals, or a finite basic open cover $D(f_i)$. We then shifted to how finite generation lets us import linear algebra into module theory. We proved the adjugate identity and Cayley–Hamilton theorem over arbitrary commutative rings by a universal specialization argument from the generic matrix, and used the determinant trick to extend Cayley–Hamilton to endomorphisms of finitely generated modules satisfying $\phi(M) \subseteq IM$. The main application was Nakayama's Lemma in several forms: if $IM = M$ for a finitely generated module then a suitable element of I acts like the identity, and if $I \subseteq \mathrm{Jac}(R)$ then $IM = M$ forces $M = 0$; in a local ring this says $\mathfrak{m}M = M$ implies $M = 0$. We used these results to check generation and surjectivity modulo $\mathfrak{m}$, compute minimal numbers of generators over local rings using $M/\mathfrak{m}M$, and prove consequences such as surjective endomorphisms of finitely generated modules being automorphisms.

6.3. **Friday, May 8, 2026.** Today we finished the final Gröbner basis worksheet of the course. We reviewed leading terms, initial ideals, division, and normal forms, emphasizing again that if G is a Gröbner basis then $f \in \langle G \rangle$ if and only if the remainder on division by G is zero, and in this case the normal form is independent of all choices. We introduced minimal and reduced Gröbner bases, explained how to pass from an arbitrary Gröbner basis to a reduced one, and proved that the reduced Gröbner basis of an ideal is unique for a fixed monomial order. We then defined standard monomials, the monomials outside $\mathrm{in}_<(I)$, and proved the Standard Monomial Basis Theorem: their residue classes form a $\mathbb{K}$-basis for S/I. This let us read dimensions and quotient bases from staircase diagrams, and characterize finite-dimensional quotients by the appearance of powers of each variable in the initial ideal. Finally, we introduced S-polynomials, Buchberger's Criterion, and Buchberger's Algorithm, which tests and constructs Gröbner bases by checking pairwise leading-term cancellations. We used these tools for hand computations, ideal membership, inclusion and equality tests, radical membership via the Rabinowitsch trick, and verification in *Macaulay2*. **Expect a quiz next Tuesday.**

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