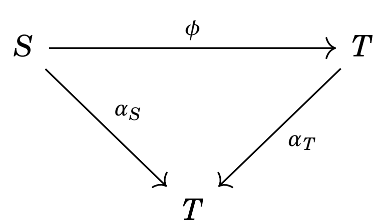
WORKSHEET 7.1: INTEGRAL EXTENSION PT. I

Throughout this course, "ring" means *commutative* ring with unity and all ring homomorphisms preserve unity. In the previous worksheets we have repeatedly moved between rings, modules, quotients, localization, and spectra. The goal of this worksheet is to introduce a language that lets us treat many ring maps as geometric families: the language of *R-algebras*. Once this language is in place, we will distinguish two finiteness notions that sound similar but behave very differently: finite generation as an algebra and finite generation as a module.

The main bridge between these two notions is ntegrality. Integral elements are elements of an extension ring that satisfy monic polynomial equations over the base. The monic condition is what lets one convert algebraic equations into module-theoretic finiteness. This is one of the basic mechanisms behind many finiteness theorems in commutative algebra and algebraic geometry.

Let R be a ring. An *R-algebra* is a ring S together with a specified ring homomorphism $\alpha : R \longrightarrow S$. Note we do not require that α be injective, however, the case when α is an inclusion, i.e., $\mathbb{K}$ into $\mathbb{K}[x]$, is generally the motivating example. We often think of an R-algebra structure on a ring S as giving us a way to multiply elements of S by elements of R, i.e., $r \cdot s := \alpha(r)s$. Thus every R-algebra S is naturally an R-module. The key difference between an R-module M and an R-algebra S is that for module we may only multiple elements of M by elements of R, but for an R-algebra we every element of S may be multiplied together. The key analogy might be: R-modules are to vector spaces, as R-algebras are to rings.

An R-algebra homomorphism $\phi : S \to T$ is a ring homomorphism compatible with the structure maps from R, meaning that the diagram



commutes. Equivalently, $\phi(\alpha_S(r)) = \alpha_T(r)$ for every $r \in R$. An *R-subalgebra* of S is a subring $S' \subset S$, which contains $\alpha(R)$. Given any set $\mathscr{S} \subset S$ the *R-algebra generated by $\mathscr{S}$* is the smallest R-subalgebra of S containing $\mathscr{S}$. We often write $R[\mathscr{S}]$ to denote R-subalgebra generated by $\mathscr{S}$, because it can explicitly be written down as the set of polynomials in the elements in $\mathscr{S}$ with coefficients in R.

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(1) **First Encounters with R-algebras.** Let $\alpha : R \to S$ be an R-algebra.

(a) Verify that the rule $r \cdot s = \alpha(r)s$ makes S into an R-module.

(b) Give an example where α is not injective. In your example, identify a nonzero element of R that acts as zero on S.

(c) Show that every quotient R/I is naturally an R-algebra.

(d) Show that if S and T are R-algebras and $\phi : S \to T$ is an R-algebra homomorphism, then ϕ is R-linear for the underlying R-module structures.

(e) Let $\mathbb{K} \subseteq \mathbb{E}$ be a field extension. Explain why $\mathbb{E}$ is a $\mathbb{K}$-algebra. What is the underlying $\mathbb{K}$-module structure?

(2) **Polynomial Rings and Generation of R-algebras.** Let S be an R-algebra and let $s_1, \ldots, s_n \in S$.

(a) Construct an R-algebra homomorphism

$$\mathrm{ev}_{s_1,\ldots,s_n} : R[x_1,\ldots,x_n] \longrightarrow S, \qquad f(x_1,\ldots,x_n) \longmapsto f(s_1,\ldots,s_n).$$

(b) Show that the image of this map is the smallest R-subalgebra of S containing $s_1,\ldots,s_n$. We denote this image by $R[s_1,\ldots,s_n]$.

(c) Prove that $S = R[s_1,\ldots,s_n]$ if and only if $\mathrm{ev}_{s_1,\ldots,s_n}$ is surjective.

(d) Deduce that S is generated as an R-algebra by n elements if and only if $S \cong R[x_1,\ldots,x_n]/I$ as an R-algebra for some ideal $I \subset R[x_1,\ldots,x_n]$.

(3) **Examples of R-algebras.** In each case, identify a finite set of algebra generators over the indicated base ring. Try to choose a minimal-looking set, though you do not need to prove minimality.

(a) $R[x]/\langle x^2 \rangle$ over R.

(b) $R[x,y]/\langle xy \rangle$ over R.

(c) R/I over R.

(d) $\mathbb{K}[t,t^{-1}]$ over $\mathbb{K}$.

(e) $\mathbb{Z}[i]$ over $\mathbb{Z}$.

(f) $\mathbb{K}[x,y]/\langle y^2 - x^3 \rangle$ over $\mathbb{K}[x]$.

(g) $\mathbb{K}(t)[[t]]$ over the $\mathbb{K}(t)$.

(h) $\mathbb{Z}[\sqrt{n}\; n \in \mathbb{Z}_{\geq 1}]$ over $\mathbb{Z}$.

As discussed above an R-algebra S is also always an R-module with the same R-multiplication. This leads to two different finiteness conditions depending on whether we wish to view S as an R-module or as an R-algebra. Let S be an R-algebra.

Definition 1. We say that S is *algebra-finite over R* if S is finitely generated s as an R-algebra.

Definition 2. We say that S is *module-finite over R* if S is finitely generated as an R-module.

The condition S be algebra-finite over R is sometimes called being *finite type* over R, i.e. we would say S is finite type over R if and only if S is algebra-finite over R. Similarly, many authors will use the phrase S

is *finite* over R or S is a *finite R-algebra* to mean S if a R-algebra that is module-finite,. In this worksheet we will attempt to restrict to using *algebra-finite* and *module-finite* to hopefully minimize confusion.

The key point is that module-finite is much stronger than algebra-finite. A module-finite algebra is controlled by finitely many R-linear combinations. An algebra-finite algebra is controlled by finitely many elements and all polynomial expressions in them.

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(4) **Module-Finite Implies Algebra-Finite.** Let S be an R-algebra and suppose that S is generated as an R-module by $m_1, \dots, m_n \in S$.

  (a) Show that the R-subalgebra $R[m_1, \dots, m_n]$ contains every R-linear combination of $m_1, \dots, m_n$.

  (b) Deduce that $S = R[m_1, \dots, m_n]$.

  (c) Conclude that every module-finite R-algebra is algebra-finite over R.

(5) **Algebra-Finite Need Not Imply Module-Finite.** Assume R is nonzero.

  (a) Show that $R[x]$ is algebra-finite over R.

  (b) Suppose $p_1, \dots, p_n \in R[x]$. Let d be the maximum of their degrees. Show that every R-linear combination of the p_i has degree at most d, unless it is zero.

  (c) Deduce that $R[x]$ is not module-finite over R. Hint: Is x^{d+1} in the R-span of $p_1, \dots, p_n$?

  (d) Specialize to $R = \mathbb{K}$ a field and restate the result in the language of vector spaces.

(6) **Permanence of Algebra-Finiteness.** The goal of this exercise is to explore how being algebra-finite plays with many natural algebraic operations: quotienting, tensoring, localizing, and composing the underlying morphisms giving the algebra structure. Consider maps of rings:

$$R \xrightarrow{\ \alpha\ } S \xrightarrow{\ \beta\ } T \ .$$

The map α gives S an R-algebra structure. On the other hand T has two different algebra structures: we may think of T as an S-algebra via the map β or think of T as an R-algebra via $\beta \circ \alpha$.

  (a) If S is algebra-finite over R and $J \subseteq S$ is an ideal, prove that S/J is algebra-finite over R.

  (b) If S is algebra-finite over R and T is algebra-finite over S, prove that T is algebra-finite over R.

  (c) Give an example where α and β are injective and T is algebra-finite over R, but S is not algebra-finite over R. Hint: Let $R = \mathbb{K}$, and $T = \mathbb{K}[x, y]$, what happens if $S = \mathbb{K}[x, xy, xy^2, xy^3, \dots]$?

  (d) If S is algebra-finite over R, and $R \to T$ is a ring map, prove that $T \otimes_R S$ is algebra-finite over T. What are algebra generators?

  (e) If $U \subset R$ is multiplicatively closed and S is algebra-finite over R, prove that $U^{-1}S$ is algebra-finite over $U^{-1}R$.

(7) **Permanence of Module-Finiteness.** The goal of this exercise is to explore how being module-finite plays with many natural algebraic operations: quotienting, tensoring, localizing, and composing the underlying morphisms giving the algebra structure. Consider maps of rings:

$$R \xrightarrow{\alpha} S \xrightarrow{\beta} T \ .$$

The map α gives S an R-module structure. On the other hand T has two different module structures: we may think of T as an S-module via the map β or think of T as an R-module via $\beta \circ \alpha$.

(a) If S is module-finite over R and $J \subseteq S$ is an ideal, prove that S/J is module-finite over R.

(b) If S is module-finite over R and T is module-finite over S, prove that T is module-finite over R.

(c) If S is module-finite over R, and $R \to T$ is a ring map, prove that $T \otimes_R S$ is module-finite over T.

(d) If $U \subset R$ is multiplicatively closed and S is module-finite over R, prove that $U^{-1}S$ is module-finite over $U^{-1}R$.

(8) **Comparing the Two Notions in Examples.** Decide whether each algebra is algebra-finite and whether it is module-finite over the indicated base ring. Justify your answers.

(a) $\mathbb{K}[x]$ over $\mathbb{K}$.

(b) $\mathbb{K}[x]/\langle x^2 \rangle$ over $\mathbb{K}$.

(c) $\mathbb{Z}[i]$ over $\mathbb{Z}$.

(d) $\mathbb{Z}[1/2]$ over $\mathbb{Z}$.

(e) $\mathbb{K}[t]$ over $\mathbb{K}[t^2]$.

(f) $\mathbb{K}[t, t^{-1}]$ over $\mathbb{K}[t]$.

Let S be an R-algebra and let $s \in S$. We say that s is *integral over* R if there exists a monic polynomial with coefficients in R that s satisfies. In other words, s is integral over R if there are elements $a_1, \dots, a_n \in R$ such that

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

in S, where each a_i acts through the structure map $R \to S$. The word *monic* is essential. For example, every rational number satisfies some nonzero polynomial over $\mathbb{Z}$, but only the integers satisfy monic polynomial equations over $\mathbb{Z}$. The monic condition is exactly what allows us to solve for high powers of s in terms of lower powers.

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(9) **First Examples of Integral Elements.** Let S be an R-algebra.

(a) Show that every element in the image of $R \to S$ is integral over R.

(b) Show that every nilpotent element of S is integral over R.

(c) Let $S = R[x]/\langle x^2 - r \rangle$ for some $r \in R$. Show that the image of x in S is integral over R.

(d) Show that $i \in \mathbb{Z}[i]$ is integral over $\mathbb{Z}$.

(e) Show that $t \in \mathbb{K}[t]$ is integral over $\mathbb{K}[t^2]$.

(f) Show that $\frac{1}{2}(\sqrt{-3}+1) \in \mathbb{Q}(\sqrt{-3})$ is integral over $\mathbb{Z}[\sqrt{-3}]$.

(g) Show directly that $1/2 \in \mathbb{Q}$ is not integral over $\mathbb{Z}$. Hint: Suppose $(1/2)^n + a_1(1/2)^{n-1} + \cdots + a_n = 0$ and clear denominators.

(10) **Integral Elements Give Finite Modules.** Let S be an R-algebra and let $s \in S$.

(a) Suppose that the element $s \in S$ is integral over R, and

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0$$

with $a_i \in R$. Show that every power s^m with $m \geq n$ is an R-linear combination of $1, s, \ldots, s^{n-1}$.

(b) Deduce that $R[s]$ is generated as an R-module by $1, s, \ldots, s^{n-1}$.

(c) Find a finite set of R-module generators for $S = \mathbb{K}[t]$ over $R = \mathbb{K}[t^2]$

(d) Find a finite set of R-module generators for $S = \mathbb{Z}[i]$ over $R = \mathbb{Z}$.

(11) **A Fundamental Criterion.** Let S be an R-algebra and let $s \in S$. Prove that the following are equivalent.

(a) s is integral over R.

(b) $R[s]$ is module-finite over R.

(c) There exists a finitely generated R-submodule $M \subseteq S$ such that $1 \in M$ and $sM \subseteq M$.

(12) **Integral Elements Form a Subring.** Let $R \to S$ be an R-algebra. Let $\overline{R}^S$ denote the set of elements of S that are integral over R. When R is an integral domain and S is a field we call $\overline{R}^S$ the *integral closure of R in S*. The *integral closure* or *normalization* of a domain R is the integral closure of R in its field of fractions $S = \operatorname{Frac}(R)$. We denote the integral closure of R by $\overline{R}$.

(a) If $x \in S$ is integral over R, prove that $R[x]$ is module-finite over R.

(b) If $x, y \in S$ are integral over R, prove that $R[x,y]$ is module-finite over R. Hint: First show $R[x]$ is finite over R, then show $R[x,y]$ is finite over $R[x]$.

(c) Let $s \in R[x,y]$. Show that multiplication by s defines an R-linear map

$$\mu_s : R[x,y] \longrightarrow R[x,y], \qquad f \longmapsto sf.$$

Use Cayley–Hamilton, or the determinant trick, to prove that s is integral over R.

(d) Conclude that $x+y$, xy, and $-x$ are integral over R.

(e) Conclude that $\overline{R}^S$ is a subring of S containingnthe image of R.

(13) **Computations with Integrality.** Let $\mathbb{K}$ be a field.

(a) Prove that the elements of $\mathbb{Q}$ integral over $\mathbb{Z}$ are exactly the integers. Hint: Write a rational number in lowest terms.

(b) Show that $\mathbb{K}[t]$ is module-finite over $\mathbb{K}[t^2,t^3]$. Find explicit module generators.

(c) Show that $\mathbb{K}[x,y]/\langle y^2-x^3\rangle$ is module-finite over $\mathbb{K}[x]$. Find explicit module generators.

(d) Let $S=\mathbb{K}[x,x^{-1}]$, viewed as a $\mathbb{K}[x]$-algebra. Show that x^{-1} is not integral over $\mathbb{K}[x]$.

(e) Deduce that $\mathbb{K}[x,x^{-1}]$ is algebra-finite but not module-finite over $\mathbb{K}[x]$.

(14) **UFDs Are Integrally Closed.** Let R be a UFD with fraction field $S=\operatorname{Frac}(R)$.

(a) Explain how we can think of S as an R-algebra?

(b) Let $\alpha\in S$. Explain why we can write $\alpha=\frac{a}{b}$ with $a,b\in R$, $b\neq 0$, and a and b having no common irreducible factors.

(c) Give an example of a domain R that is *not* a UFD for which part (b) is false.

(d) Suppose $\alpha=a/b\in S$ is integral over R and

$$\left(\frac{a}{b}\right)^n+r_1\left(\frac{a}{b}\right)^{n-1}+\cdots+r_{n-1}\left(\frac{a}{b}\right)+r_n=0$$

for $r_1,\ldots,r_n\in R$. By clearing denominators show that b divides a^n.

(e) Show that if a and b have no common irreducible factor and $b\mid a^n$, then b is a unit. Hint: Use unique factorization.

(f) Conclude that $\alpha=a/b\in R$. Therefore every element of K that is integral over R already lies in R.

(g) Conclude that every UFD is integrally closed.

(h) Show that $R=\mathbb{K}[t^2,t^3]$ is not integrally closed by finding an explicitly integral element in $S=\operatorname{Frac}(R)=\mathbb{K}(t)$ not contained in R.

(i) Show that $\mathbb{Z}[\sqrt{-3}]$ is not integrally closed. What does this mean about it being a UFD?

An R-algebra S is called an *integral extension* of R if every element of S is integral over R. Said differently, the structure map $R\to S$ is integral if all elements of the target satisfy monic polynomial equations over the source. Integral extensions are the precise setting where algebra generation and module generation interact especially well. The exercises below prove the fundamental comparison theorem:

$$\boxed{\text{module-finite}}\quad\Longleftrightarrow\quad\boxed{\text{algebra-finite and integral}}.$$

More explicitly, the exercises that follow will guide us through the proof of the following theorem.

Theorem 3. *Let $R\to S$ be an R-algebra. The R-algebra S is module-finite over R if and only if it S is algebra-finite as an R-algebra.*

Note you will often, especially in algebraic geometry, see this expressed as finite is equivalent to finite type and integral. However, again we try to avoid this language for now.

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(15) **Module-Finite Extensions Are Integral.** Let S be a module-finite R-algebra.

(a) Fix $s \in S$. Show that multiplication by s defines an R-linear map

$$\mu_s : S \longrightarrow S \qquad \text{given by} \qquad x \longmapsto sx.$$

(b) Apply the Cayley-Hamilton theorem (explain why you can apply it) to μ_s to deduce that s is integral over R.

(c) Conclude that every module-finite R-algebra is integral over R.

(16) **Finite Type Integral Extensions Are Module-Finite.** Suppose $S = R[s_1,\ldots,s_n]$ and each s_i is integral over R.

(a) Prove that $R[s_1]$ is module-finite over R.

(b) Prove that $R[s_1,s_2]$ is module-finite over $R[s_1]$, and hence module-finite over R.

(c) Continue by induction to prove that S is module-finite over R.

(d) Deduce that if S is algebra-finite and integral over R, then S is module-finite over R.

(17) **The Comparison Theorem.** Let S be an R-algebra. Prove that the following are equivalent.

(a) S is module-finite over R.

(b) S is algebra-finite over R and integral over R.

Then revisit Problem 9 and explain how the theorem classifies each example.

(18) **Permanence of Integral Extensions.** Let $R \to S \to T$ be maps of rings.

(a) Prove that every quotient R/I is integral over R.

(b) Suppose S is integral over R and T is integral over S. Prove that T is integral over R. Hint: If $t \in T$ satisfies a monic equation over S, first adjoin to R the finitely many coefficients appearing in that equation.

(c) If S is integral over R and $J \subseteq S$ is an ideal, prove that S/J is integral over $R/(J \cap R)$.

(d) If S is integral over R and $U \subset R$ is multiplicatively closed, prove that $U^{-1}S$ is integral over $U^{-1}R$.

(e) More generally, if S is integral over R and $R \to R'$ is any ring map, prove that $R' \otimes_R S$ is integral over R'.

(19) **A First Consequence for Maximal Ideals.** Let $A \subseteq B$ be an integral extension of domains.

(a) Suppose B is a field. Prove that A is a field. Hint: If $0 \neq a \in A$, then $a^{-1} \in B$ is integral over A; write a monic equation for a^{-1} and multiply by a suitable power of a.

(b) Let $R \to S$ be an integral ring map and let $\mathfrak{q} \in \operatorname{Spec}(S)$ with contraction $\mathfrak{p} = \mathfrak{q} \cap R$. Show that $S/\mathfrak{q}$ is integral over $R/\mathfrak{p}$.

(c) Deduce that if $\mathfrak{q}$ is maximal, then $\mathfrak{p}$ is maximal.

(d) Prove the converse: if $\mathfrak{p}$ is maximal, then $\mathfrak{q}$ is maximal. Hint: $S/\mathfrak{q}$ is an integral domain integral over the field $R/\mathfrak{p}$.

(e) Conclude that under an integral ring map, a prime ideal upstairs is maximal if and only if its contraction downstairs is maximal.

(20) **More Examples of Integral Extensions.** Decide whether each map is integral. If it is integral and algebra-finite, give module generators.

(a) $\mathbb{Z} \to \mathbb{Z}[i]$.

(b) $\mathbb{K}[t^2] \to \mathbb{K}[t]$.

(c) $\mathbb{K}[t^2, t^3] \to \mathbb{K}[t]$.

(d) $\mathbb{K}[x] \to \mathbb{K}[x, x^{-1}]$.

(e) $\mathbb{K}[x] \to \mathbb{K}[x,y]/\langle y^2 - x^3 \rangle$.

(f) $\mathbb{K}[x^2, xy, y^2] \to \mathbb{K}[x,y]$. Hint: Show that x and y are integral over the source.

We now introduce a numerical invariant of a ring. A *chain of prime ideals of length d* in a ring R is a strictly increasing chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$$

of prime ideals. Thus the length counts the number of strict inclusions, not the number of prime ideals appearing in the chain. For a nonzero ring R, the *Krull dimension* of R is

$$\dim(R) := \sup\{d \in \mathbb{Z}_{\geq 0} \mid \text{there exists a chain } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d \text{ in } \operatorname{Spec}(R)\}.$$

If there are chains of arbitrarily large finite length, we write $\dim(R) = \infty$. The zero ring has no prime ideals; different authors use different conventions for its dimension, so we will avoid using the zero ring in basic dimension computations unless a convention is specified. We will eventually prove the following theorem, which will turn out to be key to computing the dimension of most rings.

Theorem 4. *If $\mathbb{K}$ is a field then* $\dim \mathbb{K}[x_1, x_2, \ldots, x_n] = n$.

Proving that the dimension of the polynomial ring $\mathbb{K}[x_1, x_2, \ldots, x_n]$ is at least n is relatively easy, all one must do is find an explicit chain of prime ideals of length n. You will do this in the exercises below. Proving the equality in the theorem requires us to prove the other inequality, i.e. we must show there is no chain of prime ideals of length more than n. This turns out to be surprisingly difficult. For the remainder of the worksheet you may assume this theorem unless it trivializes an exercise.

You will also prove that if R is any ring then $\dim R[x] \geq \dim R + 1$, again by exhibiting an explicit chain of prime ideals. Intuition tells us that this should be an equality, however, this turns out to be false unless we put additional hypotheses on R, namely that R be Noetherian. This highlights a somewhat tedious fact about dimension theory in commutative algebra: unless one assumes some form of finiteness condition (Noetherian, finitely generated $\mathbb{K}$-algebra, integral, etc.) there are many pathologies.

Using the correspondence between prime ideals and irreducible closed subsets of $\operatorname{Spec}(R)$, thee intuition is that longer chains of prime ideals correspond to longer chains of irreducible closed subsets in $\operatorname{Spec}(R)$, ordered in the opposite direction.

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(21) **Unpacking the Definition.** Let R be a nonzero ring.

(a) Show that $\dim(R) = 0$ if and only if every prime ideal of R is maximal.

(b) Show that every field has Krull dimension 0.

(c) Show that if R is an integral domain that is not a field, then $\dim(R) \geq 1$.

(d) Let $I \subseteq R$ be an ideal. Use the correspondence between prime ideals of R/I and prime ideals of R containing I to describe $\dim(R/I)$ in terms of chains of prime ideals in R.

(e) Prove that $\dim(R) = \dim(R_{\text{red}})$, where $R_{\text{red}} = R/\operatorname{Nil}(R)$.

(22) **Dimension Zero Examples.** Compute the Krull dimension of each ring.

(a) $\mathbb{K}$.

(b) $\mathbb{K} \times \mathbb{K}$.

(c) $\mathbb{K}[\epsilon]/\langle \epsilon^2 \rangle$.

(d) $\mathbb{Z}/12\mathbb{Z}$.

(e) $\mathbb{K}[x]/\langle x^n \rangle$ for $n \geq 1$.

(f) $\mathbb{K}[x,y]/\langle x,y \rangle$.

(23) **Dimension One Examples.**

(a) Prove that $\dim(\mathbb{Z}) = 1$.

(b) More generally, prove that every principal ideal domain that is not a field has Krull dimension 1.

(c) Prove that $\dim(\mathbb{K}[x]) = 1$.

(d) Compute $\dim(\mathbb{K}[x]/\langle f \rangle)$ for a nonzero nonunit polynomial $f \in \mathbb{K}[x]$.

(e) Compute $\dim(\mathbb{Z}[1/n])$ for an integer $n \geq 2$.

(24) **Building Longer Chains.** Let $\mathbb{K}$ be a field.

(a) Prove that
$$\langle 0\rangle \subsetneq \langle x_1\rangle \subsetneq \langle x_1,x_2\rangle \subsetneq \cdots \subsetneq \langle x_1,\dots,x_n\rangle$$
is a chain of prime ideals in $\mathbb{K}[x_1,\dots,x_n]$.

(b) Deduce that $\dim(\mathbb{K}[x_1,\dots,x_n]) \geq n$.

(c) Let R be any nonzero ring. Show that $\dim(R[x]) \geq \dim(R)+1$ whenever $\dim(R)$ is finite. Hint: Extend a chain in R to $R[x]$, then add the ideal generated by the final prime together with x.

(25) **Dimension and Quotients.** Let $\mathbb{K}$ be a field.

(a) Compute $\dim(\mathbb{K}[x,y]/\langle xy\rangle)$. Hint: Every prime ideal containing xy contains x or y.

(b) Compute $\dim(\mathbb{K}[x,y]/\langle x^2,xy\rangle)$. Hint: First pass to the reduced ring.

(c) Compute $\dim(\mathbb{K}[x,y]/\langle y-x^2\rangle)$.

(d) Find a chain of prime ideals of length 2 in $\mathbb{K}[x,y,z]/\langle xy\rangle$.

(e) Assuming $\dim(\mathbb{K}[x_1,\dots,x_n]) = n$, compute the dimension of $\mathbb{K}[x,y,z]/\langle z-y^2+x^3\rangle$.

(26) **Local Dimension.** Let $\mathfrak{p} \in \operatorname{Spec}(R)$.

(a) Recall that prime ideals of $R_{\mathfrak{p}}$ correspond to prime ideals $\mathfrak{q} \subseteq R$ with $\mathfrak{q} \subseteq \mathfrak{p}$. Use this to describe $\dim(R_{\mathfrak{p}})$ in terms of chains of prime ideals in R ending inside $\mathfrak{p}$.

(b) Compute $\dim(\mathbb{K}[x]_{\langle x\rangle})$.

(c) Compute $\dim(\mathbb{Z}_{\langle p\rangle})$ for a prime number p.

(d) Find a chain showing that $\dim(\mathbb{K}[x,y]_{\langle x,y\rangle}) \geq 2$.

(e) Assuming $\dim(\mathbb{K}[x,y]) = 2$, prove that $\dim(\mathbb{K}[x,y]_{\langle x,y\rangle}) = 2$.