# WORKSHEET 2.1: THE ZARISKI TOPOLOGY PT. I

Throughout this course, "ring" means *commutative* ring with unity.

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(1) Recall that the spectrum and maximal spectrum of a ring $R$ are the sets:

$$\operatorname{Spec}(R) := \{P \subset R \mid P \text{ is a prime ideal}\} \quad \text{and} \quad \operatorname{mSpec}(R) := \{P \subset R \mid P \text{ is a maximal ideal}\}$$

Given any subset $\mathscr{S} \subset R$ define

$$\mathbb{V}(\mathscr{S}) := \{P \in \operatorname{Spec}(R) \mid \mathscr{S} \subset P\}.$$

(a) Describe $\operatorname{Spec}(\mathbb{K})$ and $\operatorname{mSpec}(\mathbb{K})$ when $\mathbb{K}$ is a field.

(b) Describe both $\operatorname{Spec}(\mathbb{K}[x]/\langle x^3\rangle)$ and $\operatorname{mSpec}(\mathbb{K}[x]/\langle x^3\rangle)$ when $\mathbb{K}$ is a field.

(c) For an arbitrary ring $R$, compute $\mathbb{V}(\{0\})$ and $\mathbb{V}(\{1\})$.

(d) Explain the natural poset structure on $\operatorname{Spec}(R)$. Describe the poset for $\operatorname{Spec}(\mathbb{Z})$.

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A *topological space* is a set $X$ together with a collection of subsets $\tau$, which satisfies the following conditions: i) $\varnothing \in \tau$ and $X \in \tau$, ii) $\tau$ is closed under arbitrary unions, and iii) $\tau$ is closed under finite intersections. We call $\tau$ a *topology* on $X$, the elements of $\tau$ the open subsets of $X$, and a subset $Z \subset X$ is closed if and only if $X \setminus Z$ is open (i.e., $X \setminus Z \in \tau$). If $S \subset X$ is a subset, the *subspace topology* on $S$ is the topology where the open sets of $S$ are all sets of the form $S \cap U \subset S$ where $U \subset X$ is open in $X$. A map $f : X \to Y$ of topological spaces is *continuous* if and only if $f^{-1}(U) \subset X$ is open for all open subsets $U \subset Y$, i.e., the preimage of an open set is open.

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(2) **The Zariski Topology.** We will now place a topology on the set $\operatorname{Spec}(R)$, called the Zariski topology after Oscar Zariski.

(a) Prove that if $\mathscr{S}, \mathscr{T} \subset R$ are arbitrary sets then $\mathbb{V}(\mathscr{S}) \cap \mathbb{V}(\mathscr{T}) = \mathbb{V}(\mathscr{S} \cup \mathscr{T})$. Explain why your argument extends to an *arbitrary* intersection.

(b) Prove that if $\mathscr{S}, \mathscr{T} \subset R$ are arbitrary sets then $\mathbb{V}(\mathscr{S}) \cup \mathbb{V}(\mathscr{T}) = \mathbb{V}(\mathscr{S} \cdot \mathscr{T})$ where

$$\mathscr{S} \cdot \mathscr{T} := \{st \mid s \in \mathscr{S}, t \in \mathscr{T}\}.$$

(c) Prove that the sets of the form $\mathbb{V}(\mathscr{S})$ are the closed sets of a topology on $\operatorname{Spec}(R)$. We call this topology the Zariski topology on $\operatorname{Spec}(R)$. (Hint: Rephrase the definition of a topology in terms of closed sets instead of open sets.)

 

(d) Take a break and Google Oscar Zariski and read about his life.

(3) Describe the closed subsets of $\mathrm{mSpec}(R) \subset \mathrm{Spec}(R)$ in the subspace topology.

(4) Let $\mathbb{K}$ be an algebraically closed field. This exercise will describe the Zariski topology on $\mathrm{Spec}(\mathbb{K}[x])$.

(a) Describe the points of $\mathrm{Spec}(\mathbb{K}[x])$. (Hint: There are two types.)

(b) Prove that if $\mathscr{S} \subset \mathbb{K}[x]$ then $\mathbb{V}(\mathscr{S}) = \mathbb{V}(\{f\})$ where $f$ is the greatest common divisor of all elements of $\mathscr{S}$.

(c) Show that the subspace topology on $\mathrm{mSpec}(\mathbb{K}[x])$ is the finite complement topology. (Recall the *finite complement topology* on a set $X$ is defined by letting the open sets be the empty set and sets of the form $X \setminus T$ for $T \subset X$ a subset of finite cardinality.)

(5) **Dense Points.** If $X$ is a topological space we say that a set $S \subset X$ is *dense* if $U \cap S \neq \varnothing$ for every non-empty open subset $U \subset X$.

(a) Let $p$ be a point in a topological space $X$. Prove that $\{p\}$ is a dense subset of $X$ if and only if the smallest closed subset which contains $\{p\}$ is $X$. We call such points *dense points* of $X$. (Note: no point in $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean topology is dense in this sense.)

(b) Given a prime ideal $P \subset R$ we will frequently write $[P]$ to denote the corresponding point in $\mathrm{Spec}(R)$. Prove that a point $[P] \in \mathrm{Spec}(R)$ is dense if and only if $\mathbb{V}(P) = \mathrm{Spec}(R)$.

(c) Characterize all of the dense points in the following:

(i) $\mathrm{Spec}(\mathbb{K}[x])$

(ii) $\mathrm{Spec}(\mathbb{Z})$

(iii) $\mathrm{Spec}(\mathbb{Z}[t])$

(iv) $\mathrm{Spec}(\mathbb{K}[x]/\langle x^2\rangle)$

(v) $\mathrm{Spec}(\mathbb{K}[x,y])$

(vi) $\mathrm{Spec}(\mathbb{Z}/\langle 6\rangle)$.

(vii) $\mathrm{Spec}(\mathbb{Z}/\langle 4\rangle)$.

(viii) $\mathrm{Spec}(\mathbb{Z}/\langle 12\rangle)$.

(d) Prove that if $R$ is a domain then $\mathrm{Spec}(R)$ has a unique dense point.

(e) Repeat part 5c replacing Spec with mSpec in the subspace topology. Can $\mathrm{mSpec}(R)$ ever have a dense point?

(f) Consider the following two properties (which together are implied by the Hausdorff condition) for a topological space $X$: i) for every $p \in X$ the set $\{p\}$ is closed and ii) given two distinct points $p, q \in X$ there exist open subsets $U, V \subset X$ such that $p \in U$ and $q \in V$ and $U \cap V = \varnothing$. Prove that if $X$ is a topological space with a dense point, and $X$ is not a single point, then $X$ fails both parts of this definition.

(6) Let $\mathscr{S} \subset R$ be any subset. Prove that $\mathbb{V}(\mathscr{S}) = \mathbb{V}(\langle\mathscr{S}\rangle)$ where $\langle\mathscr{S}\rangle$ denotes the ideal generated by $\mathscr{S}$.

(7) In this exercise we wish to understand the Zariski topology on $\mathrm{Spec}(\mathbb{C}[x,y])$. Fix a point $p = (a,b) \in \mathbb{C}^2$.

(a) Use the *evaluation-at-p* map: $\mathrm{ev} : \mathbb{C}[x,y] \to \mathbb{C}$ to prove that the set of polynomial functions vanishing at $p$ is a maximal ideal $\mathfrak{m}_p \subset \mathbb{C}[x,y]$.

(b) Find generators for the ideal $\mathfrak{m}_p$. Is this ideal principal?

(c) Prove that the following map is injective:

$$\begin{aligned} \mathbb{C}^2 &\xrightarrow{\ \eta\ } \operatorname{Spec}(\mathbb{C}[x,y]) \\ p &\longmapsto \mathfrak{m}_p \end{aligned}.$$

(d) Show that the map $\eta$ from part 7c is not surjective by finding at least two distinct points in $\operatorname{Spec}(\mathbb{C}[x,y])$ not in the image of $\eta$.

(e) Describe the closed subsets of $\operatorname{Spec}(\mathbb{C}[x,y])$. (Hint: They come in four types)

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A function $f : X \to Y$ of topological spaces is *continuous* if and only if $f^{-1}(U) \subset X$ is open for all open subsets $U \subset Y$, i.e., the preimage of an open is open. A *homeomorphism* $f : X \to Y$ is a continuous map of topological spaces with a continuous inverse $f^{-1} : Y \to X$. A useful fact is that a continuous map $f : X \to Y$ is a homeomorphism if and only if it is bijective and $f(U) \subset Y$ is open for every open subsets $U \subset X$.

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(8) **Ring Maps Induce Continuous Maps.** Let $\phi : R \to S$ be a ring homomorphism. The goal of this exercise is to show $\phi$ induces a map $\phi^\# : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$, which is continuous in the Zariski topology.

(a) Prove that if $[P] \in \operatorname{Spec}(S)$, then $[\phi^{-1}(P)] \in \operatorname{Spec}(R)$.

(b) Prove that the map below is continuous in the Zariski topology:

$$\begin{aligned} \operatorname{Spec}(S) &\xrightarrow{\ \phi^\#\ } \operatorname{Spec}(R) \\ [P] &\longmapsto [\phi^{-1}(P)] \end{aligned}$$

(c) For the ring map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ sending each integer to its residue class modulo 2, describe the induced map on spectra explicitly.

(d) For the ring map $\mathbb{Z} \hookrightarrow \mathbb{Q}$, describe the induced map on spectra explicitly.

(e) If $\phi : R \hookrightarrow S$ is an injective ring map, give a simple description of the induced map on spectra.

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A *category* is a collection of objects together with, for each pair of objects, a set of morphisms between them, equipped with an associative composition law and an identity morphism for each object. A *functor* between categories $\mathbf{C}$ and $\mathbf{D}$ is a mapping that assigns to each object of $\mathbf{C}$ an object of $\mathbf{D}$ and to each morphism in $\mathbf{C}$

a morphism in **D**, preserving identity morphisms and composition. A functor is *contravariant* if it reverses the order of composition and *covariant* if it preserves the order of composition.

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(9) **Spec as a Functor.** In this problem we package together much of this worksheet to define the Spec functor. Consider the map:

$$\mathbf{comRing} \longrightarrow \mathbf{Top}$$

$$R \longmapsto \mathrm{Spec}(R)$$

from the category of commutative rings with unit with ring homomorphisms to the category of topological spaces with continuous maps.

(a) Explain using Question 8 how this mapping defines a contravariant functor from **comRing** to **Top**. To check functoriality, verify

$$(\mathrm{Id}_R)^{\#} = \mathrm{Id}_{\mathrm{Spec}(R)} \qquad \text{and} \qquad (\psi \circ \phi)^{\#} = \phi^{\#} \circ \psi^{\#}$$

for ring maps $\phi : R \to S$ and $\psi : S \to T$. We call this the Spec functor.

(b) Do isomorphic rings have Spectra that are homeomorphic topological spaces? Explain.

(c) If two rings have homeomorphic spectra, are they necessarily isomorphic rings? Explain.

(10) Find a homeomorphism between $\mathrm{Spec}(\mathbb{C}[x])$ and $\mathrm{Spec}(\mathbb{C}[x,y]/\langle x-y\rangle)$. Construct the corresponding ring map explicitly in *Macaulay2* and compute its kernel.

(11) **Closed Embeddings I.** Let $I \subset R$ be an ideal and $\pi : R \to R/I$ the natural quotient map. Consider the induced map on spectra

$$\mathrm{Spec}(R/I) \xrightarrow{\pi^{\#}} \mathrm{Spec}(R)$$

In this exercise we will show this map is a closed embedding. A *closed embedding* is a map of topological spaces whose image is closed and which is a homeomorphism onto its image.

(a) Using the correspondence between ideals in $R$ and $R/I$ give an explicit description of $\pi^{\#}$.

(b) Prove that the image of $\pi^{\#}$ is equal to $\mathbb{V}(I)$.

(c) Explain why the ideal correspondence implies that $\pi^{\#}$ is injective.

(d) Prove that $\pi^{\#}$ is a closed embedding onto $\mathbb{V}(I)$, in the subspace topology, by showing that if $J \subset R/I$ is an ideal then

$$\pi^{\#}(\mathbb{V}(J)) = \mathbb{V}\left(\pi^{-1}(J)\right).$$