Worksheet 1.2: Quotients, Primes, Maximal Ideals

Let R be any ring. In this course, "ring" means *commutative ring with unity.*

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(1) **Operations on Ideals.** Throughout this problem let I and J be ideals in a ring R.
  (a) Prove or disprove that $I \cap J$ is an ideal.
  (b) Prove or disprove that $I \cup J$ is an ideal.
  (c) Prove or disprove that the product of ideals, defined below, is an ideal:
$$IJ \coloneqq \left\{ a_1 b_1 + \cdots + a_t b_t \;\middle|\; a_i \in I \;\; b_i \in J, \;\text{ for all } t \right\}.$$
  (d) Prove or disprove that the sum of ideals, defined below, is an ideal:
$$I + J \coloneqq \left\{ a + b \;\middle|\; a \in I \;\; b \in J \right\}.$$
  (e) Prove or disprove that the radical of an ideal, defined below, is an ideal:
$$\sqrt{I} \coloneqq \left\{ a \in R \;\middle|\; a^k \in I \text{ for some } k \in \mathbb{Z}_{\geq 1} \right\}.$$

(2) Let $\mathbb{K}$ be a field. The rings $\mathbb{Z}$, $\mathbb{K}[x]$, and $\mathbb{K}[[t]]$ are all principal ideal domains (PIDs) meaning domains where every ideal is generated by a single element. If $I = \langle a \rangle$ and $J = \langle b \rangle$ give, with proof, a careful concrete description of generators for IJ, $I \cap J$, $I + J$, and $\sqrt{I}$ in each of these rings.

Universal Properties. Throughout this course we will see numerous constructions and operations (e.g., quotients, tensor products, localizations, completions, etc.) each of these will have an explicit construction where we describe the objects precisely as sets. However, there is an alternative perspective, which is just as useful, describing these objects via their *universal properties*, which roughly asks what makes each object special amongst all other such objects. The standard framework for a universal property is, "Object X is unique up to isomorphism among all objects having property P."

(3) **Universal Property of Quotients.** Let R be a ring and $I \subseteq R$ an ideal. Let $\pi\colon R \to R/I$ be the quotient map. Suppose $\varphi\colon R \to S$ is a ring homomorphism such that $I \subseteq \ker(\varphi)$. Define a function
$$\widetilde{\varphi}\colon R/I \to S \qquad \text{by} \qquad \widetilde{\varphi}(r + I) = \varphi(r).$$
  (a) Prove that $\widetilde{\varphi}$ is well-defined.
  (b) Prove that $\widetilde{\varphi}$ is a ring homomorphism.
  (c) Prove that $\widetilde{\varphi} \circ \pi = \varphi$.
  (d) Prove that $\widetilde{\varphi}$ is the unique ring homomorphism $R/I \to S$ with the property that
$$\widetilde{\varphi} \circ \pi = \varphi.$$
  (e) Explain in words what this says about the quotient ring R/I.

(4) In this problem, we wish to consider the evaluation map:
$$\mathrm{ev}_i\colon \mathbb{Z}[x] \to \mathbb{C}, \qquad f(x) \mapsto f(i)$$
  (a) Prove that ev_i is a ring homomorphism.

(b) Show that $\langle x^2 + 1\rangle \subset \ker(\mathrm{ev}_i)$.
(c) Use Problem 3 to produce a ring homomorphism

$$\widetilde{\mathrm{ev}}_i \colon \mathbb{Z}[x]/\langle x^2 + \rangle \to \mathbb{C}.$$

(d) What does $\widetilde{\mathrm{ev}}_i$ do to the class of x in $\mathbb{Z}[x]/\langle x^2 + 1\rangle$?
(e) Prove that $\mathbb{Z}[x]/\langle x^2 + 1\rangle$ is isomorphic to $\mathbb{Z}[i]$. Is $\langle x^2 + 1\rangle$ prime and/or maximal?

(5) **Universal Property of Polynomial Rings.** The goal of this exercise is to prove the following characterization of polynomial rings. Let $\varphi\colon R \to S$ be a ring homomorphism. For any element $s \in S$ there is a unique ring homomorphism $\psi\colon R[x] \to S$ such that $\psi(x) = s$ and the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, hook] & S \\ R[x] \arrow[ur, dashed, "\psi"'] & \end{tikzcd}.$$

(a) Prove there is at most one ring homomorphism $\psi : R[x] \to S$ with the above properties.
(b) Construct such a homomorphism explicitly by setting

$$\psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \varphi(a_i) s^i.$$

(c) Conclude that to give a ring homomorphism $R[x] \to S$ is equivalent to giving:
- a ring homomorphism $R \to S$, and
- an element of S.

(6) **Evaluation maps.** Let R be a ring and let $a \in R$. Define the evaluation map

$$\mathrm{ev}_a \colon R[x] \to R, \qquad f(x) \mapsto f(a).$$

(a) Prove that ev_a is a surjective ring homomorphism.
(b) Prove that $\ker(\mathrm{ev}_a) = \langle x - a\rangle$.
(c) Use Problem 1 to show that ev_a factors through a ring homomorphism

$$R[x]/(x - a) \to R.$$

(d) Explain what this means if R is a field or R is an integral domain.
(e) Give an example of a ring R and prime/maximal ideal $I \subset R[x]$ such that $I \neq \ker(\mathrm{ev}_a)$ for all $a \in R$.