

WORKSHEET 1.2: QUOTIENTS AND PRIME AND MAXIMAL IDEALS

Throughout this course, "ring" means *commutative* ring with unity. A general theme throughout commutative algebra is that when given a special type of ideals, you should often ask whether it can be classified by morphisms in some way. In this worksheet we study quotient rings, which gives us a way to classify all ideals in terms of homomorphisms, i.e., ideals are precisely the subsets of rings that arise as kernels. We saw one direction of this claim on a previous exercise, the other is the following very important theorem.

Theorem 1. *Let R be a ring and $I \subset R$ an ideal.*

- (1) *There exists a ring, which we denote R/I , together with a surjective ring homomorphism $\pi : R \rightarrow R/I$ such that $\ker(\pi) = I$. Moreover, R/I and π are unique up to isomorphism.*
- (2) *There is a inclusion-preserving bijection:*

$$\begin{array}{ccc} \{ \text{ideals in } R/I \} & \xrightarrow{\sim} & \left\{ \begin{array}{l} \text{ideals in } R \\ \text{containing } I \end{array} \right\} \\ J & \longmapsto & \pi^{-1}(J) \end{array}$$

You have hopefully seen a proof of this theorem where you give an explicit construction of R/I as a set of equivalence classes determined by I . Later in this worksheet we will see how we can also determine R/I by its universal property.

Two important types of ideals that will appear throughout this course are prime and maximal ideals. A proper ideal $P \subset R$ is *prime* if whenever $ab \in P$ we have $a \in P$ or $b \in P$. A proper ideal $I \subset R$ is *maximal* if it is maximal among proper ideals of R with respect to inclusion. Put differently, a proper ideal $I \subset R$ is maximal if and only if given a proper ideal $J \subset R$ such that $I \subset J$ then $J = I$.

Theorem 2. *Let $I \subset R$ be an ideal. Then I is prime if and only if R/I is an integral domain, and I is maximal if and only if R/I is a field.*

Recall a non-zero ring R is an *integral domain*, often shortened to just *domain*, if R has no zero-divisors. A field is a ring R such that every non-zero element of R is a unit. An important equivalent characterization is that a ring R is a field if and only if the only ideals of R are $\langle 0 \rangle$ and R . If $\mathfrak{m} \subset R$ is a maximal ideal the *residue field* of \mathfrak{m} is the field R/\mathfrak{m} .

Note the above theorem says that both prime and maximal ideals can be characterized by morphisms in the sense: i) An ideal $I \subset R$ is prime if and only if there exists a surjective morphism $\phi : R \rightarrow S$ where S is an integral domain and $I = \ker(\phi)$; and ii) An ideal $I \subset R$ is maximal if and only if there exists a surjective morphism $\phi : R \rightarrow \mathbb{K}$ where \mathbb{K} is a field and $I = \ker(\phi)$. Since every domain can be viewed as a subring of a field, a fact we shall prove later in the course, we can also rephrase (i) and (ii) above as: An ideal $I \subset R$

is prime if and only if there exists a morphism $\phi: R \rightarrow \mathbb{K}$ where \mathbb{K} is a field and $I = \ker(\phi)$. Moreover, I is maximal if and only if one can find ϕ which is surjective.

(1) **Operations on Ideals.** Throughout this problem let I and J be ideals in a ring R .

(a) Prove or disprove that $I \cap J$ is an ideal.

(b) Prove or disprove that $I \cup J$ is an ideal.

(c) Prove or disprove that the product of ideals, defined below, is an ideal:

$$IJ := \{a_1b_1 + \cdots + a_tb_t \mid a_i \in I, b_i \in J, \text{ for all } t\}.$$

(d) Prove or disprove that the sum of ideals, defined below, is an ideal:

$$I + J := \{a + b \mid a \in I, b \in J\}.$$

(e) Prove or disprove that the radical of an ideal, defined below, is an ideal:

$$\sqrt{I} := \{a \in R \mid a^k \in I \text{ for some } k \in \mathbb{Z}_{\geq 1}\}.$$

(2) Let \mathbb{K} be a field. The rings \mathbb{Z} , $\mathbb{K}[x]$, and $\mathbb{K}[[t]]$ are all principal ideal domains (PIDs) meaning domains where every ideal is generated by a single element. If $I = \langle a \rangle$ and $J = \langle b \rangle$ give, with proof, a careful concrete description of generators for IJ , $I \cap J$, $I + J$, and \sqrt{I} in each of these rings.

Throughout this course we will see numerous constructions and operations (e.g., quotients, tensor products, localizations, completions, etc.) each of these will have an explicit construction where we describe the objects precisely as sets. However, there is an alternative perspective, which is just as useful, describing these objects via their *universal properties*, which roughly asks what makes each object special amongst all other such objects. The standard framework for a universal property is, “Object X is unique up to isomorphism among all objects having property P.”

(3) **Universal Property of Quotients.** Let R be a ring and $I \subseteq R$ an ideal. Let $\pi: R \rightarrow R/I$ be the quotient map. Suppose $\varphi: R \rightarrow S$ is a ring homomorphism such that $I \subseteq \ker(\varphi)$. Define a function

$$\tilde{\varphi}: R/I \rightarrow S \quad \text{by} \quad \tilde{\varphi}(r + I) = \varphi(r).$$

(a) Prove that $\tilde{\varphi}$ is well-defined ring homomorphism.

(b) Prove that $\tilde{\varphi} \circ \pi = \varphi$.

(c) Prove that $\tilde{\varphi}$ is the unique ring homomorphism $R/I \rightarrow S$ with the property that

$$\tilde{\varphi} \circ \pi = \varphi.$$

(d) Explain in words what this says about the quotient ring R/I .

(4) In this problem, we wish to consider the evaluation map:

$$\text{ev}_i: \mathbb{Z}[x] \rightarrow \mathbb{C}, \quad f(x) \mapsto f(i)$$

(a) Prove that ev_i is a ring homomorphism.

(b) Show that $\langle x^2 + 1 \rangle \subset \ker(\text{ev}_i)$.

(c) Use Question 3 to produce a ring homomorphism

$$\widetilde{\text{ev}}_i: \mathbb{Z}[x]/\langle x^2 + 1 \rangle \rightarrow \mathbb{C}.$$

(d) What does $\widetilde{\text{ev}}_i$ do to the class of x in $\mathbb{Z}[x]/\langle x^2 + 1 \rangle$?

(e) Prove that $\mathbb{Z}[x]/\langle x^2 + 1 \rangle$ is isomorphic to $\mathbb{Z}[i]$. Is $\langle x^2 + 1 \rangle$ prime and/or maximal?

(5) **Universal Property of Polynomial Rings.** The goal of this exercise is to prove the following characterization of polynomial rings. Let $\varphi: R \rightarrow S$ be a ring homomorphism. For any element $s \in S$ there is a unique ring homomorphism $\psi: R[x] \rightarrow S$ such that $\psi(x) = s$ and the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & \nearrow \psi & \\ R[x] & & \end{array} .$$

(a) Prove there is at most one ring homomorphism $\psi: R[x] \rightarrow S$ with the above properties.

(b) Construct such a homomorphism explicitly by setting

$$\psi\left(\sum_{i=0}^n a_i x^i\right) = \sum_{i=0}^n \varphi(a_i) s^i.$$

(c) Conclude that to give a ring homomorphism $R[x] \rightarrow S$ is equivalent to giving:

- a ring homomorphism $R \rightarrow S$, and
- an element of S .

(6) **Evaluation maps.** Let R be a ring and let $a \in R$. Define the evaluation map

$$\text{ev}_a: R[x] \rightarrow R, \quad f(x) \mapsto f(a).$$

(a) Prove that ev_a is a surjective ring homomorphism and $\ker(\text{ev}_a) = \langle x - a \rangle$.

(b) Use Question 3 to show that ev_a factors through a ring homomorphism $R[x]/\langle x - a \rangle \rightarrow R$.

(c) Explain what this means if R is a field or R is an integral domain.

(d) Give an example of a ring R and prime/maximal ideal $I \subset R[x]$ such that $I \neq \ker(\text{ev}_a)$ for all $a \in R$.