Worksheet 1.1: Basics of Rings

Let R be any ring. In this course, "ring" means *commutative ring with unity.*

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(1) Let R be a ring and consider $R[[t]]$ the ring of formal power series with coefficients in R. The goal of this exercise is prove that:

$$(R[[t]])^\times = \left\{ u + tf(t) \;\middle|\; u \in R^\times,\;\; f(t) \in R[[t]] \right\}.$$

(a) Elements of $R[[t]]$ can be written as $f = \sum_{k=0}^{\infty} f_k t^k$ for $f_k \in R$. We call f_0 the constant (or lead) term of f. If $f = \sum_{k=0}^{\infty} f_k t^k$ and $g = \sum_{k=0}^{\infty} g_k t^k$ then find a formula for the lead term of fg.
(b) Using the formula from (a) prove that if f is a unit then $f_0 \in R^\times$.
(c) Find a formula for the coefficient of t^k appearing in fg.
(d) Assume that $f_0 \in R^\times$. Prove that f is a unit by explicitly constructing an element $g \in R[[t]]$ such that $fg = 1$. (Hint: the condition that $fg = 1$ places constraints on each coefficient of fg. Use the constraints together with part (c) to recursively solve for the coefficients of g.)

(2) (a) Prove that an element $u \in R$ is a unit if and only if the principal ideal $\langle u \rangle = R$.
(b) Prove that if ab and b are units, then a is a unit.
(c) Prove that if u is a unit and $a \in R$, then a and ua generate the same principal ideal.
(d) In the ring $\mathbb{Z}/\langle 12 \rangle$, determine which elements are units.

(3) Given two rings R and S we let $\mathrm{Hom}_{\mathbf{Ring}}(R, S)$ be the set of ring homomorphism from R to S. When there is little room for confusion we will often drop the subscript **Ring** and simply write $\mathrm{Hom}(R, S)$.
(a) Prove or disprove: If R and S are rings, then $\mathrm{Hom}(R, S)$ is a ring under point-wise multiplication and point-wise addition.
(b) Give an explicit description of the hom-sets below:
  (i) $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$
  (ii) $\mathrm{Hom}(\mathbb{Z}/\langle 2 \rangle, \mathbb{Z})$
  (iii) $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/\langle 2 \rangle)$
  (iv) $\mathrm{Hom}(\mathbb{Z}, \mathbb{C}[x])$
(c) Based on part (b) state and prove a characterization of $\mathrm{Hom}(\mathbb{Z}, R)$ for any ring R.

(4) Let $\phi : R \to S$ be a ring homomorphism.
(a) Prove that if $J \subset S$ is an ideal then $\phi^{-1}(J) \subset R$ is an ideal.
(b) Assuming ϕ is injective, give a nice description of $\phi^{-1}(J)$ for an ideal $J \subset S$.
(c) Find a counterexample showing that if $I \subset R$ is an ideal $\phi(I)$ need not be an ideal.
(d) Prove that if ϕ is surjective then $\phi(I)$ is an ideal of S.
(e) Let $I \subset R$ be an ideal. Prove that if $\ker(\phi) \subset I$ then $\phi^{-1}(\phi(I)) = I$?
(f) Is the assumption that $\ker(\phi) \subset I$ needed in part (e)?