THE QUANTITATIVE BEHAVIOR OF ASYMPTOTIC SYZYGIES FOR HIRZEBRUCH SURFACES

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We study the quantitative behavior of asymptotic syzygies for certain toric surfaces, including Hirzebruch surfaces. In particular, we show that the asymptotic linear syzygies of Hirzebruch surfaces embedded by $\mathcal{O}(d, 2)$ conform to Ein, Erman, and Lazarsfeld's normality heuristic. We also show that the higher degree asymptotic syzygies are not asymptotically normally distributed.

In [5] Ein, Erman, and Lazarsfeld proposed a heuristic for the quantitative behavior of asymptotic syzygies: a Betti table that is "sufficiently positive" behaves approximately like the Betti table of a large Koszul complex. In particular, the entries of each row of such a Betti table should, after possibly rescaling, look like a normal Gaussian distribution. The goal of this note is to consider Ein, Erman, and Lazarsfeld's normality heuristic for a new class of examples, namely certain toric surfaces (including Hirzebruch surfaces) when the embedding line bundle grows in a semi-ample fashion.

Our main results are twofold, and can be visualized in the case of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d, 2)$. Below we plot the entries of the two interesting rows of the Betti table for $d = 3, 5, 10,$ and 20.

In line with Ein, Erman, and Lazarsfeld's normality heuristic Figure 1 shows that the first row of these Betti tables appears to be approaching a normal distribution. Our first result says that this behavior generalizes to all Hirzebruch surfaces, which we denote by $\mathbb{F}_t$, with respect to the line bundle
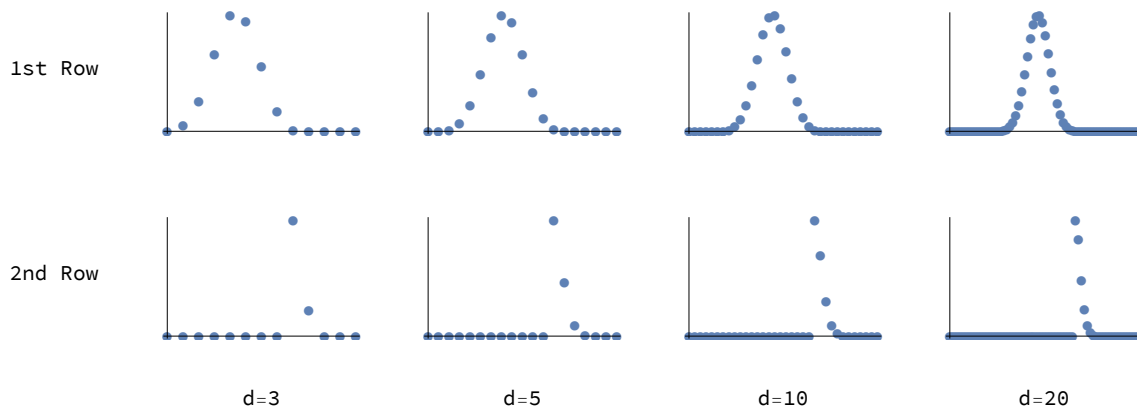

Figure 1. Entries of the first and second rows of the Betti table of $\mathbb{P}^1 \times \mathbb{P}^1$.

$\mathbb{O}_{\mathbb{F}_t}(d, 2) = 2E + dF$ where we identify $\text{Nef}(\mathbb{F}_t) \cong \mathbb{N}\langle E \rangle \oplus \mathbb{N}\langle F \rangle$ with F being the fiber of the map to $\mathbb{P}^1$ and E being the exceptional curve. Note the only previously known examples satisfying Ein, Erman, and Lazarsfeld's heuristic were smooth curves [5, Proposition A] and random monomial ideals [6, Theorem 1.4].

Returning to Figure 1, one sees that the second row of the Betti table of $\mathbb{P}^1 \times \mathbb{P}^1$ embedding by $\mathbb{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d, 2)$ looks quite dissimilar to a normal distribution. At best, the second row appears to converge to some fraction of a normal distribution. Our second main result shows that this failure of Ein, Erman, and Lazarsfeld's normality heuristic occurs for the second row of all Hirzebruch surfaces embedded by $\mathbb{O}_{\mathbb{F}_t}(d, 2)$.

As far as we are aware, this is the first known set of examples where something other than Ein, Erman, and Lazarsfeld's heuristic describes the quantitative behavior of asymptotic syzygies. We emphasize, however, that this is not a counterexample to [5, Conjecture B] because that conjecture assumes the embedding line bundle grows in an ample fashion, and our examples are based on semi-ample growth.

To prove these theorems we build upon work of Lemmens who provided formulas for the graded Betti numbers for certain toric surfaces in terms of invariants of the associated polytopes. Our key observation is that by proving general results concerning the convergence of binomial distributions to normal distributions we can use these formulas to examine Ein, Erman, and Lazarsfeld's normality heuristic.

Turning to the details, let X be a projective variety of dimension n over an arbitrary field $\Bbbk$. Given a sequence of very ample line bundles $\{L_d\}_{d \in \mathbb{N}}$, we wish to study how the graded Betti numbers of X behave asymptotically with respect to L_d for $d \gg 0$. That is, we are interested in the syzygies of the section ring

$$R(X; L_d) := \bigoplus_{k \in \mathbb{Z}} H^0(X, k \cdot L_d),$$

as a module over $S = \text{Sym } H^0(X, L_d) \cong \Bbbk[x_0, x_1, \ldots, x_{r_d}]$. Considering the graded minimal free resolution

$$0 \longleftarrow R(X; L_d) \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \qquad \cdots \longleftarrow F_{r_d} \longleftarrow 0,$$

we let

$$K_{p,q}(X; L_d) := \text{span}_\Bbbk \langle \text{minimal generators of } F_p \text{ of degree } (p + q) \rangle$$

be the finite-dimensional $\Bbbk$-vector space of minimal syzygies of homological degree p and degree $p + q$. With this notation, F_p is isomorphic to $\bigoplus_q K_{p,q}(X; L_d) \otimes_\Bbbk S(-(p + q))$. We write $k_{p,q}(X; L_d)$ for $\dim K_{p,q}(X; L_d)$, and then form the Betti table of $(X; L_d)$ by placing $k_{p,q}(X; L_d)$ in the (q, p)-th spot as shown below:

	0	1	$\cdots$	$\cdots$	r_d
0	$k_{0,0}$	$k_{1,0}$	$\cdots$		$k_{r_d,0}$
1	$k_{0,1}$	$k_{1,1}$	$\cdots$		$k_{r_d,1}$
$\vdots$	$\vdots$	$\vdots$			$\vdots$
n	$k_{0,n}$	$k_{1,n}$	$\cdots$		$k_{r_d,n}$

In this setup we may more precisely state Ein, Erman, and Lazarsfeld's heuristic as follows: if $\{L_d\}_{d \in \mathbb{N}}$ is a sequence of line bundles growing in positivity, then for any $q \in [1, n]$ there exists a function $F_q(d)$,

depending on X, such that if $\{p_d\}_{d\in\mathbb{N}}$ is a sequence of nonnegative integers such that

$$\text{(1)} \qquad \lim_{d\to\infty}\left(p_d - \left(\frac{r_d}{2} + a\frac{\sqrt{r_d}}{2}\right)\right) = 0,$$

where $a \in \mathbb{R}$ is a fixed constant, then

$$F_q(d) \cdot k_{p_d,q}(X; L_d) \to e^{-a^2/2}.$$

Notice that the assumption that the sequence $\{p_d\}_{d\in\mathbb{N}}$ satisfies (1) is crucial. In particular, this change of coordinates is necessary for binomial distributions to converge to normal distributions. As we will use it frequently, we take the time to label it here.

Assumption. We say that a sequence $\{p_d\}_{d\in\mathbb{N}}$ satisfies assumption ★ if, for some real number $a \in \mathbb{R}$,

$$\lim_{d\to\infty}\left(p_d - \left(\frac{r_d}{2} + a\frac{\sqrt{r_d}}{2}\right)\right) = 0. \qquad\qquad ★$$

We now state our main results concerning Ein, Erman, and Lazarsfeld's normality heuristic for Hirzebruch surfaces, which we denote by $\mathbb{F}_t$, embedded by the line bundle $\mathcal{O}_{\mathbb{F}_t}(d, 2)$.

Theorem A. *If $\{p_d\}_{d\in\mathbb{N}}$ is a sequence of nonnegative integers satisfying ★, then*

$$\frac{3\sqrt{2\pi}}{2^{r_d}\sqrt{r_d}} \cdot k_{p_d,1}(\mathbb{F}_t, \mathcal{O}_{\mathbb{F}_t}(d, 2)) = e^{-a^2/2}\left(1 + O\left(\frac{1}{\sqrt{r_d}}\right)\right).$$

Theorem B. *There does not exist a function $F_2(d)$ such that if $\{p_d\}_{d\in\mathbb{N}}$ is a sequence of nonnegative integers satisfying ★, then*

$$F_2(d) \cdot k_{p_d,2}(\mathbb{F}_t, \mathcal{O}_{\mathbb{F}_t}(d, 2)) = e^{-a^2/2}\left(1 + O\left(\frac{1}{\sqrt{r_d}}\right)\right).$$

Notice that in this setting we are considering a slightly weaker positivity condition than was initially considered in [5, Conjecture B]. In particular, the embedding line bundle is not growing in an ample fashion, but instead in a semi-ample fashion. This failure of Ein, Erman, and Lazarsfeld's heuristic when $q = 2$ is likely related to the fact that the nonvanishing of asymptotic syzygies in the setting of semi-ample growth is quite nuanced. See for example [1], where the author shows that the nonvanishing of asymptotic syzygies for products of projective spaces is not necessarily described by the nonvanishing theorems of Ein and Lazarsfeld [4].

The note is structured as follows: In Section 1 we study the asymptotic distribution of graded Betti numbers for a family of toric surfaces. Section 1 also includes the proofs of Theorems A and B. Section 2 contains technical results used in the proofs in Section 1.

1. Asymptotic normality for certain toric surfaces

In this section we consider Ein, Erman, and Lazarsfeld's normality heuristic for certain toric surfaces, and prove slight generalizations of Theorems A and B. Specifically we consider the toric surface X_δ whose

associated normal fan $\Sigma_\delta \subset \mathbb{R}^2$ has four cones given by the rays $\{(1,0), (0,1), (0,-1), (-2,\delta)\}$, where $\delta \in \mathbb{N}$.
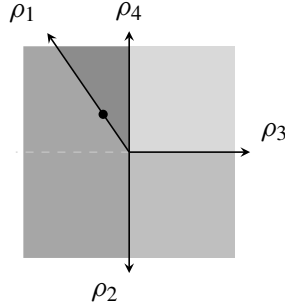
When δ is even, X_δ is isomorphic to the Hirzebruch surface $\mathbb{F}_{\delta/2}$. However, when δ is odd, X_δ is singular, with two $\mathbb{Z}/2\mathbb{Z}$-singularities [2, Proposition 10.1.2].

For each ray $\rho_1, \rho_2, \rho_3, \rho_4$, there is a corresponding prime torus invariant divisor $D_{\rho_1}, D_{\rho_2}, D_{\rho_3}, D_{\rho_4}$, which may be thought of as the irreducible components of $X_\delta \setminus T$, where $T \subset X_\delta$ is the torus. When δ is even, so that $X_\delta \cong \mathbb{F}_{\delta/2}$, these divisors are related to the generators of $\mathrm{Nef}(\mathbb{F}_{\delta/2})$ described in the introduction as follows: $D_{\rho_1} \sim D_{\rho_3} \sim F$, $D_{\rho_2} \sim E + (\delta/2)F$, and $D_{\rho_4} \sim E$. In particular, $\mathrm{Nef}(\mathbb{F}_{\delta/2})$ is generated by D_{ρ_1} and D_{ρ_2}.

We are interested in the syzygies of X_δ with respect to the divisor $L_d = dD_{\rho_1} + 2D_{\rho_2}$ when δ is even and $L_d = 2dD_{\rho_1} + 2D_{\rho_2}$ when δ is odd. The corresponding polytope for these divisors is

$$\Delta_d = \mathrm{conv}\{(0,0),\ (d,0),\ (0,2),\ (d+\delta,2)\}.$$

For example, if $\delta = 3$ and $d = 2$ then Δ_d is the polytope below:
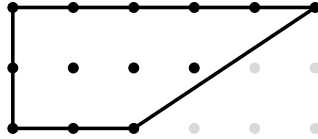
Throughout the remainder of the paper, we write n_d for $\#(\Delta_d \cap \mathbb{Z}^2)$ and $n_d^{(1)}$ for $\#(\mathrm{int}(\Delta_d) \cap \mathbb{Z}^2)$. For $d \geq 1$ the divisor L_d is very ample and defines an embedding $X_\delta \to \mathbb{P}H^0(L_d) \cong \mathbb{P}^{r_d}$ where $r_d = n_d - 1$. A straightforward argument using Pick's theorem shows that

$$r_d = 3d + C_\delta \quad \text{and} \quad n_d^{(1)} = \tfrac{1}{3}r_d + E_\delta, \qquad \text{where} \quad C_\delta := \tfrac{3}{2}\delta + \tfrac{1}{2}\gcd(\delta,2) + 1 \quad \text{and} \quad E_\delta := \tfrac{1}{3}\gcd(\delta,2) - \tfrac{1}{3}.$$

We now prove analogues of Theorems A and B for $(X_\delta; L_d)$. As X_δ is isomorphic to $\mathbb{F}_{\delta/2}$ when δ is even, Theorems A and B follow from these slightly more general theorems about $(X_\delta; L_d)$.

First, we show that Ein, Erman, and Lazarsfeld's normality heuristic accurately describes the quantitative behavior of the asymptotic linear syzygies of $(X_\delta; L_d)$.

Theorem 1. *Set $F_1(d) = \dfrac{3\sqrt{2\pi}}{2^{r_d}\sqrt{r_d}}$. If $\{p_d\}_{d\in\mathbb{N}}$ is a sequence of nonnegative integers satisfying ★ then*

$$F_1(d) \cdot k_{p_d,1}(X_\delta; L_d) = e^{-a^2/2}\left(1 + O\left(\frac{1}{\sqrt{r_d}}\right)\right).$$

Proof. Using [9, Corollary 5] together with the fact that $n_d = r_d + 1$ and $n_d^{(1)} = \frac{1}{3}r_d + E_\delta$, we know that

$$k_{p_d,1}(X_\delta; L_d) = \max\{p_d - \tfrac{2}{3}r_d + E_\delta + 1, \, 0\}\binom{r_d - 2}{p_d - 1} + p_d\binom{r_d}{p_d + 1} - (\tfrac{4}{3}r_d + E_\delta - 1)\binom{r_d - 2}{p_d - 1}.$$

By assumption ★, we may replace p_d with $\frac{1}{2}r_d + \frac{1}{2}a\sqrt{r_d}$, giving

$$(2) \quad F_1(d) \cdot k_{p_d,1}(X_\delta; L_d)$$

$$\sim F_1(d)\left(\max\left\{-\frac{r_d}{6} + a\frac{\sqrt{r_d}}{2} + E_\delta + 1, \, 0\right\}\binom{r_d - 2}{p_d - 1} + \left(\frac{r_d}{2} + a\frac{\sqrt{r_d}}{2}\right)\binom{r_d}{p_d + 1} - (\tfrac{4}{3}r_d + E_\delta - 1)\binom{r_d - 2}{p_d - 1}\right).$$

Proposition 3 implies that for any constants $c_1, c_2 \in \mathbb{Z}$ and $c_3 \in \mathbb{R}$ both $F_1(d) \cdot a\frac{\sqrt{r_d}}{2} \cdot \binom{r_d + c_1}{p_d + c_2}$ and $F_1(d) \cdot c_3 \cdot \binom{r_d + c_1}{p_d + c_2}$ tend to zero as $d \to \infty$. Hence we may ignore these terms in the above line, and rewrite (2) as

$$F_1(d) \cdot k_{p_d,1}(X_\delta; L_d) \sim F_1(d) \cdot \left(\max\{-\tfrac{r_d}{6}, \, 0\}\binom{r_d - 2}{p_d - 1} + \frac{r_d}{2}\binom{r_d}{p_d + 1} - \frac{4}{3}r_d\binom{r_d - 2}{p_d}\right)$$

$$\sim \frac{3\sqrt{2\pi r_d}}{2^{r_d + 1}}\binom{r_d}{p_d + 1} - \frac{8\sqrt{2\pi r_d}}{2^{r_d + 1}}\binom{r_d - 2}{p_d - 1}.$$

The result now follows directly from Proposition 3. $\qquad\square$

Our second theorem in this section shows that the higher degree asymptotic syzygies of $(X_\delta; L_d)$ do not behave as suggested by Ein, Erman, and Lazarsfeld's normality heuristic. In particular, the entries in the $q = 2$ row of the Betti table of $(X_\delta; L_d)$ do not converge to a normal distribution as $d \to \infty$.

Theorem 2. *There does not exist a function $F_2(d)$ such that if $\{p_d\}_{d \in \mathbb{N}}$ is a sequence of nonnegative integers satisfying ★ then*

$$F_2(d) \cdot k_{p_d,2}(X_\delta; L_d) = e^{-a^2/2}(1 + O(1/\sqrt{r_d})).$$

Proof. Similar to the proof of Theorem 1, by using [9, Corollary 5], the fact that $n_d = r_d + 1$ and $n_d^{(1)} = \frac{1}{3}r_d + E_\delta$, and assumption ★ we see that

$$k_{p_d,2}(X_\delta; L_d) = \max\left\{\frac{-r_d}{6} + a\frac{\sqrt{r_d}}{2} + E_\delta + 2, \, 0\right\}\binom{r_d - 2}{p_d}.$$

However, $\frac{-r_d}{6} + a\frac{\sqrt{r_d}}{2} + E_\delta + 3 < 0$ for $d \gg 0$, and so for any function $F_2(d)$ we have that

$$F_2(d) \cdot k_{p_d,2}(X_\delta; L_d) = F_2(d) \cdot \max\left\{\frac{-r_d}{6} + a\frac{\sqrt{r_d}}{2} + E_\delta + 3, \, 0\right\}\binom{r_d - 2}{p_d} = 0. \qquad\square$$

Remark. Theorems 1 and 2 only depend on the values of $k_{p,q}(X_\delta; L_d)$ for p in a neighborhood of $\frac{1}{2}r_d$, and hence are related in part to the study of Green's N_p-property [10; 8]. In particular, both theorems may also be deduced from Schenck's work showing that $k_{p,2}(X_\delta; L_d) = 0$ for all $p \leq \#(\partial\Delta_d \cap \mathbb{Z}^2) - 3$ [10, Corollary 2.1]. Since in the cases we are considering Δ_d is height two, $\#(\partial\Delta_d \cap \mathbb{Z}^2)$ is relatively large compared to $\frac{1}{2}r_d$. Using this one can deduce Theorem 2. One may then deduce Theorem 1 by noting that this vanishing implies that $k_{p,1}(X_\delta; L_d)$ for p around $\frac{1}{2}r_d$ can be deduced from the Hilbert function of $R(X_\delta; L_d)$. We thank the referee for pointing out this alternative argument.

2. Technical results

Here we gather a series of technical results crucial to the proofs of our main theorems. The key result of this section is the following special case of the local central limit theorem. While standard in many probability texts, see [3, Theorem 3.5.3], we take the time to prove it as this version of the local de Moivre–Laplace theorem, with precise constants and error terms, is crucial to proving our main theorems.

Proposition 3. *Suppose $\{r_d\}_{d \in \mathbb{N}}$ is a sequence such that $r_d \to \infty$ as $d \to \infty$. If there exists a sequence $\{p_d\}_{d \in \mathbb{N}}$ of nonnegative integers satisfying ★ then, for any constants $c_1, c_2 \in \mathbb{Z}$,*

$$\frac{\sqrt{2\pi r_d}}{2^{r_d+1}} \binom{r_d+c_1}{p_d+c_2} = 2^{c_1} e^{-a^2/2} \big(1 + O(1/\sqrt{r_d})\big).$$

Before proving this proposition we need a lemma.

Lemma 4. *Suppose $\{r_d\}_{d \in \mathbb{N}}$ is a sequence such that $r_d \to \infty$ as $d \to \infty$. For any $a \in \mathbb{R}$,*

$$(3) \qquad \left(\frac{r_d}{r_d + a\sqrt{r_d}}\right)^{r_d/2 + a\sqrt{r_d}/2} \left(\frac{r_d}{r_d - a\sqrt{r_d}}\right)^{r_d/2 - a\sqrt{r_d}/2} = e^{-a^2/2}\big(1 + O(1/\sqrt{r_d})\big).$$

Proof. Let Q denote the left-hand side of (3). We have

$$(4) \qquad -\log Q = \frac{r_d + a\sqrt{r_d}}{2} \log\left(1 + \frac{a}{\sqrt{r_d}}\right) + \frac{r_d - a\sqrt{r_d}}{2} \log\left(1 - \frac{a}{\sqrt{r_d}}\right).$$

Using the Taylor expansion of $\log(1 + x)$ the right-hand side of (4) may be rewritten as

$$(5) \qquad \frac{r_d + a\sqrt{r_d}}{2}\left(\frac{a}{\sqrt{r_d}} - \frac{a^2}{2r_d} + O(r_d^{-3/2})\right) + \frac{r_d - a\sqrt{r_d}}{2}\left(-\frac{a}{\sqrt{r_d}} - \frac{a^2}{2r_d} - O(r_d^{-3/2})\right).$$

Simplifying (5) and combining it with (4) shows that

$$(6) \qquad \log Q = -\frac{a^2}{2} + O\big(1/\sqrt{r_d}\big).$$

The result now follows by exponentiating both sides $\qquad \square$

Proof of Proposition 3. Throughout the proof we write $\tilde{c}$ for $c_1 - c_2$. A straightforward computation shows that $O(r_d)$, $O(r_d + c_1)$, $O(p_d + c_2)$ and $O(r_d - p_d + \tilde{c})$ are all equal, and so we will not distinguish between them. Using Stirling's formula for $n!$ we see that

$$(7) \quad \frac{\sqrt{2\pi r_d}}{2^{r_d+1}} \binom{r_d+c_1}{p_d+c_2} = 2^{-(r_d+1)}\sqrt{\frac{r_d^2 + c_1 r_d}{(p_d+c_2)(r_d-p_d+\tilde{c})}}$$

$$\times \left(\frac{r_d+c_1}{p_d+c_2}\right)^{c_2}\left(\frac{r_d+c_1}{r_d-p_d+\tilde{c}}\right)^{\tilde{c}}\left(\frac{r_d+c_1}{p_d+c_2}\right)^{p_d}\left(\frac{r_d+c_1}{r_d-p_d+\tilde{c}}\right)^{r_d-p_d}\big(1 + O(1/r_d)\big).$$

Since c_1 and c_2 are constants and r_d and p_d tend to infinity as $d \to \infty$, using assumption ★ one can show that

$$\lim_{d \to \infty}\left(\frac{r_d+c_1}{p_d+c_2}\right)^{c_2} = 2^{c_2}, \quad \lim_{d \to \infty}\left(\frac{r_d+c_1}{r_d-p_d+\tilde{c}}\right)^{\tilde{c}} = 2^{\tilde{c}}, \quad \lim_{d \to \infty}\frac{r_d^2}{r_d p_d - p_d^2} = 4.$$

Using these limits, we see that (7) can be simplified to

$$(8) \qquad \frac{\sqrt{2\pi r_d}}{2^{r_d+1}} \binom{r_d+c_1}{p_d+c_2} = 2^{-r_d} 2^{c_1} \left(\frac{r_d+c_1}{p_d+c_2} \right)^{p_d} \left(\frac{r_d+c_1}{r_d-p_d+\tilde{c}} \right)^{r_d-p_d} \left(1 + O(1/r_d) \right).$$

Now we show that we can reduce to the case when $c_1 = c_2 = \tilde{c} = 0$. Towards this, notice that by the assumption ★, we know that $r_d - p_d$ tends to infinity as $d \to \infty$. Combining this with the fact that r_d and p_d also tend to infinity as $d \to \infty$, we see that

$$\lim_{d \to \infty} \frac{\left(\frac{r_d}{p_d} \right)^{p_d} \left(\frac{r_d}{r_d-p_d} \right)^{r_d-p_d}}{\left(\frac{r_d+c_1}{p_d+c_2} \right)^{p_d} \left(\frac{r_d+c_1}{r_d-p_d+\tilde{c}} \right)^{r_d-p_d}} = \lim_{d \to \infty} \left[\left(\frac{p_d+c_2}{p_d} \frac{r_d}{r_d+c_1} \right)^{p_d} \left(\frac{r_d}{r_d+c_1} \frac{r_d-p_d+\tilde{c}}{r_d-p_d} \right)^{r_d-p_d} \right]$$

$$= \lim_{d \to \infty} \left(1 + \frac{c_2}{p_d} \right)^{p_d} \left(1 - \frac{c_1}{r_d+c_1} \right)^{r_d} \left(1 + \frac{\tilde{c}}{r_d-p_d} \right)^{r_d-p_d}$$

$$= e^{c_2} \cdot e^{-c_1} \cdot e^{\tilde{c}} = 1.$$

Thus we may rewrite (8) as

$$(9) \qquad \frac{\sqrt{2\pi r_d}}{2^{r_d+1}} \binom{r_d+c_1}{p_d+c_2} = 2^{-r_d} 2^{c_1} \left(\frac{r_d}{p_d} \right)^{p_d} \left(\frac{r_d}{r_d-p_d} \right)^{r_d-p_d} \left(1 + O(1/r_d) \right).$$

Assumption ★ allows us to substitute $\frac{r_d}{2} + \frac{a\sqrt{r_d}}{2}$ everywhere we see p_d in (9). Doing this gives

$$(10) \qquad \frac{\sqrt{2\pi r_d}}{2^{r_d+1}} \binom{r_d+c_1}{p_d+c_2} = 2^{c_1} \left(\frac{r_d}{r_d+a\sqrt{r_d}} \right)^{r_d/2+a\sqrt{r_d}/2} \left(\frac{r_d}{r_d-a\sqrt{r_d}} \right)^{r_d/2-a\sqrt{r_d}/2} \left(1 + O(1/d) \right),$$

from which the result follows using Lemma 4. $\qquad\square$

Acknowledgements

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