The Degree of SO(n, $\mathbb{C}$)

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Abstract We provide a closed formula for the degree of SO(n, $\mathbb{C}$). In addition, we test symbolic and numerical techniques for computing the degree of SO(n, $\mathbb{C}$). As an application of our results, we give a formula for the number of critical points of a low-rank semidefinite programming problem. Finally, we provide evidence for a conjecture regarding the real locus of SO(n, $\mathbb{C}$).

MSC 2010 codes: 14L35, 20G20, 15N30

1 Introduction

The *special orthogonal group* SO(n, $\mathbb{R}$) is the group of automorphisms of $\mathbb{R}^n$ which preserve the standard inner product and have determinant equal to one. The complex special orthogonal group is the complexification of SO(n, $\mathbb{R}$) or, more explicitly, the

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G.G. Smith, B. Sturmfels (eds.), *Combinatorial Algebraic Geometry*,
Fields Institute Communications 80, https://doi.org/10.1007/978-1-4939-7486-3_11

group of matrices $\mathrm{SO}(n, \mathbb{C}) := \{M \in \mathbb{C}^{n\times n} : \det(M) = 1 \text{ and } M^{\mathsf{T}}M = I\}$. Since the defining conditions are polynomials in the entries of the matrix M, the group $\mathrm{SO}(n, \mathbb{C})$ is also a complex variety.

The degree of a complex variety $X \subset \mathbb{C}^n$ is the generic number of points in the intersection of X with a linear space of complementary dimension. Problem 4 on Grassmannians in [19] seeks a formula for the degree of $\mathrm{SO}(n, \mathbb{C})$. Our main result provides this.

Theorem 1.1 *The degree of* $\mathrm{SO}(n, \mathbb{C})$ *equals* $2^{n-1} \det \left[\binom{2n-2i-2j}{n-2i}\right]_{1\le i,j\le \lfloor n/2\rfloor}$.

Our proof of Theorem 1.1 uses a formula of Kazarnovskij [14] for the degree of the image of a representation of a connected reductive algebraic group over an algebraically closed field; see Theorem 2.4 for more information. By applying this formula to the case of the standard representation of $\mathrm{SO}(n, \mathbb{C})$, we are able to express the degree in terms of its root data and other invariants. As an added feature, Theorem 4.2 provides a combinatorial interpretation of this degree in terms of non-intersecting lattice paths. In contrast with Theorem 1.1, the combinatorial statement has the benefit of being obviously non-negative.

In order to verify Theorem 1.1, as well as explore the structure of $\mathrm{SO}(n, \mathbb{C})$ in further depth, it is useful to compute this degree explicitly. We were able to do this, for small n, using symbolic and numerical computations. A comparison of the success of these approaches is illustrated in Table 1.

Remark 1.2 Let $\mathbb{k}$ be a field of characteristic zero. We can define $\mathrm{SO}(n, \mathbb{k})$ using the same system of equations because they are defined over the prime field $\mathbb{Q}$. For a field $\mathbb{k}$ that is not algebraically closed, the degree of a variety can be defined in terms of the Hilbert series of its coordinate ring. Since the Hilbert series does not depend on the choice of $\mathbb{k}$, the degree does not either. We choose to work over $\mathbb{C}$ not only for simplicity, but also so that we may use the above definition of degree.

Remark 1.3 Our methods are not restricted to $\mathrm{SO}(n, \mathbb{C})$ and can be used to compute the degree of other algebraic groups. For example, we provide a similar closed formula for the degree of the symplectic group in Sect. 3 and a combinatorial reinterpretation in Sect. 4.

Table 1 Degree of $\mathrm{SO}(n, \mathbb{C})$ computed in various ways

n	Symbolic	Numerical	Formula
2	2	2	2
3	8	8	8
4	40	40	40
5	384	384	384
6	-	4768	4768
7	-	111616	111616
8	-	-	3433600
9	-	-	196968448

This project started in the spring of 2014, when Benjamin Recht asked the fifth author to describe the geometry of a low-rank optimization problem; see Sect. 5. In particular, Benjamin asked why the augmented Lagrangian algorithm for solving this problem [5] almost always recovers the correct optimum despite the existence of multiple local minima. It quickly became clear that to even compute the number of local extrema, one needs to know the degree of the orthogonal group. In Sect. 5, we find a formula for the number of critical points of the low-rank semidefinite programming problem; see Theorem 5.3.

The rest of this article is organized as follows. In Sect. 2, we give the reader a brief introduction to algebraic groups and state the Kazarnovskij Theorem. Section 3 proves Theorem 1.1 by applying the Kazarnovskij Theorem and simplifying the resulting expressions. After simplification, we are left with a determinant of binomial coefficients that can be interpreted combinatorially using the celebrated Gessel–Viennot Lemma; see Sect. 4. The relationship between the degree of $\mathrm{SO}(n, \mathbb{C})$ and the degree of the low-rank optimization programming problem is elaborated upon in Sect. 5. Section 6 contains descriptions of the symbolic and numerical techniques involved in the explicit computation of $\deg \mathrm{SO}(n, \mathbb{C})$. Finally, in Sect. 7, we explore questions involving the real points on $\mathrm{SO}(n, \mathbb{C})$.

2 Background

In this section, we provide the reader with the language to understand the Kazarnovskij Theorem, our main tool for determining the degree of $\mathrm{SO}(n, \mathbb{C})$. We invite those who already are familiar with Lie theory to skip to the statement of Theorem 2.4. Aside from applying Theorem 2.4, no understanding of the material in this section is necessary for understanding the remainder of the proof of Theorem 1.1. A more thorough treatment of the theory of algebraic groups can be found in [6, 8, 13].

An *algebraic group* G is a variety equipped with a group structure such that multiplication and inversion are both regular maps on G. When the unipotent radical of G is trivial and G is over an algebraically closed field, we say that G is a *reductive group*. Throughout this section, we let G denote a connected reductive algebraic group over an algebraically closed field $\mathbb{k}$. Let $\mathbb{G}_{\mathrm{m}}$ denote the multiplicative group of $\mathbb{k}$, so as a set $\mathbb{G}_{\mathrm{m}} = \mathbb{k} \setminus \{0\}$. Let T denote a fixed maximal torus of G, that is a subgroup of G isomorphic to $\mathbb{G}_{\mathrm{m}}^r$ and which is maximal with respect to inclusion. The number $r \in \mathbb{N}$ is well-defined and is called the *rank* of G. After fixing T, we define the *Weyl group* of G, denoted $W(G)$, to be the quotient of the normalizer of T by its centralizer: $W(G) := N_G(T)/Z_G(T)$. Like the rank, the group $W(G)$ does not depend on the choice of T up to isomorphism.

Example 2.1 The map $\mathrm{R}\colon \mathbb{G}_{\mathrm{m}} \to \mathrm{SO}(2,\Bbbk)$, given by $\mathrm{R}(t) := \frac{1}{2}\begin{bmatrix} t+t^{-1} & -i(t-t^{-1}) \\ i(t-t^{-1}) & t+t^{-1} \end{bmatrix}$, parametrizes $\mathrm{SO}(2,\Bbbk)$ and is a group isomorphism. If $\Bbbk = \mathbb{C}$, then the rotation by an angle θ corresponds to the matrix $\mathrm{R}(e^{i\theta})$. Therefore, the algebraic group $\mathrm{SO}(2,\Bbbk)$ has rank 1.

If $r \geq 1$, then the maximal tori of rank r in their respective algebraic groups are

$$T_{2r} := \left\{ \begin{bmatrix} \mathrm{R}(t_1) & 0 & 0 \cdots & 0 \\ 0 & \mathrm{R}(t_2) & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & 0 \cdots & \mathrm{R}(t_r) \end{bmatrix} : t_i \in \mathbb{G}_{\mathrm{m}} \right\} \cong \mathrm{SO}(2,\Bbbk)^r \subset \mathrm{SO}(2r,\Bbbk)\,,$$

$$T_{2r+1} := \left\{ \begin{bmatrix} \mathrm{R}(t_1) & 0 & 0 \cdots & 0 & 0 \\ 0 & \mathrm{R}(t_2) & 0 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & \mathrm{R}(t_r) & 0 \\ 0 & 0 & 0 \cdots & 0 & 1 \end{bmatrix} : t_i \in \mathbb{G}_{\mathrm{m}} \right\} \cong \mathrm{SO}(2,\Bbbk)^r \subset \mathrm{SO}(2r+1,\Bbbk)\,.$$

Therefore, we have $\operatorname{rank}\mathrm{SO}(2r,\Bbbk) = \operatorname{rank}\mathrm{SO}(2r+1,\Bbbk) = r$ and see that the rank of $\mathrm{SO}(n,\Bbbk)$ depends on the parity of n.

The *character group* $M(T)$ is the set of algebraic group homomorphisms from T to $\mathbb{G}_{\mathrm{m}}$. In other words, $M(T) := \mathrm{Hom}_{\mathrm{AlgGrp}}(T,\mathbb{G}_{\mathrm{m}})$ consists of the group homomorphisms defined by polynomial maps. Since T is isomorphic to $\mathbb{G}_{\mathrm{m}}^r$, all such homomorphisms must be of the form $(t_1, t_2, \ldots, t_r) \mapsto t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r}$ for some integers $a_1, a_2, \ldots, a_r$. Hence, the character group $M(T)$ is isomorphic to $\mathbb{Z}^r$ and, for this reason, it is often called the character lattice. The *group of 1-parameter subgroups* $N(T) := \mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{G}_{\mathrm{m}}, T)$ is dual to $M(T)$ and is also isomorphic to $\mathbb{Z}^r$. Indeed, each 1-parameter subgroup is of the form $t \mapsto (t^{b_1}, t^{b_2}, \ldots, t^{b_r})$ for some integers $b_1, b_2, \ldots, b_r$. Moreover, there exists a natural bilinear pairing $M(T) \times N(T) \to \mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{G}_{\mathrm{m}},\mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}$ given by $\langle \chi, \sigma\rangle \mapsto \chi \circ \sigma$.

Now, if $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G, then we attach to it special characters called weights. A *weight* of the representation ρ is a character $\chi \in M(T)$ such that the set

$$V_\chi := \bigcap_{s\in T} \ker\bigl(\rho(s) - \chi(s)\,\mathrm{Id}_V\bigr)$$

is non-trivial. This condition is equivalent to saying that all of the matrices in $\{\rho(s) : s \in T\}$ have a simultaneous eigenvector $v \in V$ such that the associated eigenvalue for $\rho(s)$ is $\chi(s)$. We write C_V for the convex hull of the weights of the representation ρ.

Example 2.2 An important example for us comes from the defining representation $\rho\colon \mathrm{SO}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(n,\mathbb{C})$. Let $e_1, e_2, \ldots, e_n$ denote the standard basis for $\mathbb{C}^n$. For any $t \in \mathbb{G}_{\mathrm{m}}$, the matrix $\mathrm{R}(t) \in \mathrm{SO}(2,\mathbb{C})$ has eigenvectors $e_1 + ie_2$ and $e_1 - ie_2$ with eigenvalues t and t^{-1} respectively. From the explicit description of the maximal torus T in Example 2.1, it follows that the eigenvectors of ρ are all vectors of the form $e_{2j-1} \pm ie_{2j}$ with $1 \le j \le r$ and the corresponding eigenvalues are $t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}$. These eigenvalues, viewed as characters, are the weights of ρ. Additionally, when $n = 2r + 1$, we see that e_{2r+1} is an eigenvector with eigenvalue 1, corresponding to the trivial character.

Another representation of a matrix group $G \subseteq \mathrm{End}(V)$ is the *adjoint representation* $\mathrm{Ad}\colon G \to \mathrm{GL}\big(\mathrm{End}(V)\big)$, where $\mathrm{Ad}(g)$ the linear map defined by $A \mapsto gAg^{-1}$. The *roots* of G are the nonzero weights of the adjoint representation. Given a linear functional ℓ on $M(T)$, we define the *positive roots* of G with respect to ℓ to be the roots χ such that $\ell(\chi) > 0$. We denote the positive roots of G by $\alpha_1, \alpha_2, \ldots, \alpha_l$. For the algebraic groups in this paper, we can choose ℓ to be the inner product with the vector $(r, r-1, \ldots, 1)$, so that a root of the form $e_j - e_k$ is positive if and only if $j < k$. To each root α, we associate a *coroot* $\check{\alpha}$, defined to be the linear function $\check{\alpha}(\mathbf{x}) := 2\langle \mathbf{x}, \alpha\rangle / \langle \alpha, \alpha\rangle$ where the pairing is $W(G)$-invariant. Throughout this paper, we fix the pairing to be the standard inner product.

Example 2.3 We now describe the roots of $\mathrm{SO}(n,\mathbb{C})$, starting with $n = 2r$. The simultaneous eigenvectors of $\mathrm{Ad}(s)$ over all $s \in T$ are matrices A with the following structure. These matrices are zero outside a (2×2)-block B in rows $2j-1, 2j$ and columns $2k-1, 2k$ for some $1 \le j, k \le r$. Furthermore, $B = v_1 v_2^{\mathsf{T}}$ with each vector v_k, for $1 \le k \le 2$, equals one of the eigenvectors of $\mathrm{R}(t)$, namely $e_1 \pm ie_2$. If $s \in T$ has blocks along the diagonal $\mathrm{R}(t_j)$ with $t_1, t_2, \ldots, t_r \in \mathbb{G}_{\mathrm{m}}$, then the matrix $\mathrm{Ad}(s)(A)$ will also be zero except in the same (2×2)-block, and that block will be $\mathrm{R}(t_j) B\, \mathrm{R}(t_k)^{\mathsf{T}} = t_j^{\pm 1} t_k^{\pm 1} B$, where the signs depend on the choices of v_1 and v_2. Taking the exponent vectors of these eigenvalues, we see that the roots of $\mathrm{SO}(2r,\mathbb{C})$ are the characters of the form $\pm(e_j \pm e_k)$ for $1 \le j, k \le r$.

When $n = 2r + 1$, the matrix A has an extra row and column. If the matrix A has support only in the last column, then we have $\mathrm{Ad}(s)(A) = sAs^{-1}$. But s^{-1} acts trivially on the left, while s acts on the last column as an element of $\mathrm{GL}(n,\mathbb{C})$ as in the standard representation. As in Example 2.2, the eigenvalues are $t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}, 1$. The same weights appear for A with support in the last row. Hence, the roots of $\mathrm{SO}(2r+1,\mathbb{C})$ are $\pm(e_j \pm e_k)$ for $1 \le j, k \le r$ and $\pm e_i$ for $1 \le i \le r$.

Associated to the algebraic group G is a Lie algebra $\mathfrak{g}$ that comes equipped with a Lie bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. A *Cartan subalgebra* $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$ that is self-normalizing; if $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$, then we have $y \in \mathfrak{h}$. Let $S(\mathfrak{h}^*)$ be the ring of polynomial functions on $\mathfrak{h}$. The Weyl group $W(G)$ acts on $\mathfrak{h}$, and this extends to an action of $W(G)$ on $S(\mathfrak{h}^*)$. The space $S(\mathfrak{h}^*)^{W(G)}$ of polynomials which are invariant up to the action of $W(G)$ is generated by r homogeneous polynomials whose degrees, $c_1 + 1, c_2 + 1, \ldots, c_r + 1$, are uniquely determined. The values $c_1, c_1, \ldots, c_r$ are called *Coxeter exponents*.

Table 2 Data required to apply the Kazarnovskij Theorem

Group	Dimension	Rank	Positive roots	Weights	$\lvert W(G)\rvert$	Coxeter exponents
$\mathrm{SO}(2r+1,\mathbb{C})$	$\binom{2r+1}{2}$	r	$\{e_i \pm e_j\}_{i<j} \cup \{e_i\}$	$\{\pm e_i\}$	$r!2^r$	$1, 3, 5, \ldots, 2r-1$
$\mathrm{Sp}(2r,\mathbb{C})$	$\binom{2r+1}{2}$	r	$\{e_i \pm e_j\}_{i<j} \cup \{2e_i\}$	$\{\pm e_i\}$	$r!2^r$	$1, 3, 5, \ldots, 2r-1$
$\mathrm{SO}(2r,\mathbb{C})$	$\binom{2r}{2}$	r	$\{e_i \pm e_j\}_{i<j}$	$\{\pm e_i\}$	$r!2^{r-1}$	$1, 3, 5, \ldots, 2r-3, r-1$

We are now prepared to state the Kazarnovskij Theorem.

Theorem 2.4 (Kazarnovskij Theorem, [6, Proposition 4.7.18]) *Let G be a connected reductive algebraic group of dimension m and rank r over an algebraically closed field. If $\rho\colon G \to \mathrm{GL}(V)$ is a representation with finite kernel, then we have*

$$\deg\left(\overline{\rho(G)}\right) = \frac{m!}{\lvert W(G)\rvert\,(c_1!c_2!\cdots c_r!)^2\,\lvert\ker(\rho)\rvert}\int_{C_V}(\check{\alpha}_1\check{\alpha}_2\cdots\check{\alpha}_r)^2\,dv\,,$$

where $W(G)$ is the Weyl group, the c_i are Coxeter exponents, C_V is the convex hull of the weights, and the $\check{\alpha}_i$ are the coroots.

If ρ is the standard representation for an algebraic group G, then it follows that $\deg\overline{\rho(G)} = \deg G$. Thus, in order to compute $\deg\mathrm{SO}(n,\mathbb{C})$, all we must do is apply this theorem for the standard representation of $\mathrm{SO}(n,\mathbb{C})$. The relevant data for this theorem is given in Table 2 for $\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{C})$.

3 Main Result: The Degree of $\mathrm{SO}(n,\mathbb{C})$

We now prove our main result, Theorem 1.1. At the end of this section, we also use the same method to obtain a formula for the degree of the symplectic group.

We begin by applying Theorem 2.4 to $\mathrm{SO}(2r,\mathbb{C})$ and $\mathrm{SO}(2r+1,\mathbb{C})$ to obtain

$$\deg\mathrm{SO}(2r,\mathbb{C}) = \frac{\binom{2r}{2}!}{r!2^{r-1}\big((r-1)!\big)^2\prod_{k=1}^{r-1}\big((2k-1)!\big)^2}\int_{C_V}\prod_{1\le i<j\le r}(x_i^2-x_j^2)^2\,dv\,,$$

$$\deg\mathrm{SO}(2r+1,\mathbb{C}) = \frac{\binom{2r+1}{2}!}{r!2^{r}\prod_{k=1}^{r}\big((2k-1)!\big)^2}\int_{C_V}\prod_{1\le i<j\le r}(x_i^2-x_j^2)^2\prod_{i=1}^{r}(2x_i)^2\,dv\,.$$

To compute the degree of $\mathrm{SO}(n,\mathbb{C})$, it suffices to find formulas for these integrals. We do this by expanding the integrand into monomials and integrating the result. We first use the well-known expression for the determinant of the Vandermonde matrix:

$$\prod_{1\le i<j\le r}(y_j-y_i) = \sum_{\sigma\in\mathfrak{S}_r}\mathrm{sgn}(\sigma)\prod_{i=1}^{r}y_i^{\sigma(i)-1}\,,$$

where $\mathfrak{S}_r$ denotes the symmetric group on $\{1, 2, \ldots, r\}$. Substituting $y_i = x_i^2$ and squaring the entire expression yields

$$\prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 = \sum_{\sigma, \tau \in \mathfrak{S}_r} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{r} x_i^{2\sigma(i)+2\tau(i)-4} . \tag{1}$$

Every variable in the integrand is being raised to an even power, and C_V is the convex hull of weights $\{\pm e_i\}$. Because of this symmetry, the integrals over C_V are 2^r times the same integrals over the r-simplex $\Delta_r := \operatorname{conv}(0, e_1, e_2, \ldots, e_r) \subset \mathbb{R}^r$. Hence, we have reduced the computation to understanding the integral of any monomial over the simplex Δ_r. The following lemma provides the required formula.

Lemma 3.1 ([15, Lemma 4.23]) *Consider the r-simplex $\Delta_r := \operatorname{conv}(0, e_1, e_2, \ldots, e_r)$ in $\mathbb{R}^r$. If $\mathbf{a} = (a_1, a_2, \ldots, a_r) \in \mathbb{Z}_{>0}^r$, then we have*

$$\int_{\Delta_r} \mathbf{x}^{\mathbf{a}} \, d\mathbf{x} = \int_{\Delta_r} x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} \, dx_1 dx_2 \cdots dx_r = \frac{1}{(r + \sum_i a_i)!} \prod_i a_i! .$$

With these preliminaries, we can now prove the key technical result in this section.

Proposition 3.2 *We have*

$$\int_{C_V} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \, dv = \frac{r! 2^r}{\binom{2r}{2}!} \det\big[(2i + 2j - 4)!\big]_{1 \le i,j \le r} ,$$

$$\int_{C_V} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 \, dv = \frac{r! 2^{3r}}{\binom{2r+1}{2}!} \det\big[(2i + 2j - 2)!\big]_{1 \le i,j \le r} .$$

Proof Exploiting the symmetry of C_v along with equation (1) gives

$$I_{\text{odd}}(r) := \int_{C_V} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 \, dv = 2^r \int_{\Delta_r} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 \, dv$$

$$= 2^{3r} \sum_{\sigma, \tau \in \mathfrak{S}_r} \operatorname{sgn}(\sigma\tau) \int_{\Delta_r} \prod_{i=1}^{r} x_i^{2\sigma(i)+2\tau(i)-2} \, dv .$$

As the integrand is homogeneous of degree $4\binom{r}{2} + 2r$, Lemma 3.1 yields

$$I_{\text{odd}}(r) = \frac{2^{3r}}{\big(4\binom{r}{2} + 3r\big)!} \sum_{\sigma, \tau \in \mathfrak{S}_r} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{r} \big(2\sigma(i) + 2\tau(i) - 2\big)! .$$

Since $\sigma \in \mathfrak{S}_r$, we may reindex the product by $\sigma^{-1}(i)$ rather than i to obtain $\prod_{i=1}^{r}\big(2\sigma(i)+2\tau(i)-2\big)! = \prod_{i=1}^{r}\big(2i+2\tau\sigma^{-1}(i)-2\big)!$. Ranging over all $\sigma, \tau \in \mathfrak{S}_r$, each permutation in $\mathfrak{S}_r$ appears exactly $r!$ times as the composition $\upsilon := \tau\sigma^{-1}$ and $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\upsilon)$. Therefore, we have

$$\begin{aligned} I_{\text{odd}}(r) &= \frac{r!2^{3r}}{\big(4\binom{r}{2}+3r\big)!}\sum_{\upsilon\in\mathfrak{S}_r}\operatorname{sgn}(\upsilon)\prod_{i=1}^{r}(2i+2\upsilon(i)-2)! \\ &= \frac{r!2^{3r}}{\binom{2r+1}{2}!}\det\big[(2i+2j-2)!\big]_{1\le i,j\le r}. \end{aligned}$$

The calculation for $I_{\text{even}}(r) := \int_{C_V}\prod_{1\le i<j\le r}(x_i^2 - x_j^2)^2 \; dv$ follows the same steps. □

Proof of Theorem 1.1 Combining Theorem 2.4, the data in Table 2, and Proposition 3.2, we have

$$\begin{aligned} \deg \mathrm{SO}(2r+1,\mathbb{C}) &= \frac{\binom{2r+1}{2}!}{r!2^r\prod_{k=1}^{r}\big((2k-1)!\big)^2}\int_{C_V}\prod_{1\le i<j\le r}(x_i^2-x_j^2)^2\prod_{i=1}^{r}(2x_i)^2\, dv \\ &= \frac{2^{2r}}{\prod_{k=1}^{r}\big((2k-1)!\big)^2}\det\big[(2i+2j-2)!\big]_{1\le i,j\le r}. \end{aligned}$$

Since the determinant is linear in each row and column, we obtain

$$\deg \mathrm{SO}(2r+1,\mathbb{C}) = 2^{2r}\det\left[\frac{(2i+2j-2)!}{(2i-1)!(2j-1)!}\right] = 2^{2r}\det\left[\binom{2i+2j-2}{2i-1}\right]_{1\le i,j\le r}.$$

Reversing the order of the rows and columns of the final matrix and reindexing produces the required formula. Similarly, for the even case, we have

$$\begin{aligned} \deg \mathrm{SO}(2r,\mathbb{C}) &= \frac{\binom{2r}{2}!}{r!2^{r-1}\big((r-1)!\big)^2\prod_{k=1}^{r-1}\big((2k-1)!\big)^2}\int_{C_V}\prod_{1\le i<j\le r}(x_i^2-x_j^2)^2\, dv \\ &= \frac{2}{\big((r-1)!\big)^2\prod_{k=1}^{r-1}\big((2k-1)!\big)^2}\det\big[(2i+2j-4)!\big]_{1\le i,j\le r} \\ &= \frac{2(2^{r-1})^2}{\prod_{k=1}^{r-1}(2k)^2\prod_{k=1}^{r-1}\big((2k-1)!\big)^2}\det\big[(2i+2j-4)!\big]_{1\le i,j\le r} \end{aligned}$$

$$= \frac{2^{2r-1}}{\prod_{k=1}^{r}\big((2k-2)!\big)^2} \det\big[(2i+2j-4)!\big]_{1\le i,j\le r}$$

$$= 2^{2r-1} \det\left[\frac{(2i+2j-4)!}{(2i-2)!(2j-2)!}\right]_{1\le i,j\le r}$$

$$= 2^{2r-1} \det\left[\binom{2i+2j-4}{2i-2}\right] = 2^{2r-1} \det\left[\binom{4r-2i-2j}{2r-2i}\right]_{1\le i,j\le r}.$$

□

Since the orthogonal group $\mathrm{O}(n,\mathbb{C}) := \{M \in \mathbb{C}^{n\times n} : M^{\mathsf{T}}M = MM^{\mathsf{T}} = I\}$ has two connected components that are isomorphic to $\mathrm{SO}(n,\mathbb{C})$, we immediately get a formula for the degree of $\mathrm{O}(n,\mathbb{C})$.

Corollary 3.3 *The degree of* $\mathrm{O}(n,\mathbb{C})$ *equals* $2^n \det\big[\binom{2n-2i-2j}{n-2i}\big]_{1\le i,j\le \lfloor \frac{n}{2}\rfloor}$.

We also easily obtain the degree of the symplectic group $\mathrm{Sp}(2r,\mathbb{C})$. By definition, we have $\mathrm{Sp}(2r,\mathbb{C}) := \{M \in \mathbb{C}^{2r\times 2r} : M^{\mathsf{T}}\Omega M = \Omega\}$ where

$$\Omega := \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix} \in \mathbb{C}^{2r\times 2r}.$$

Corollary 3.4 *We have* $\deg \mathrm{SO}(2r+1,\mathbb{C}) = 2^{2r} \deg \mathrm{Sp}(2r,\mathbb{C})$ *and*

$$\deg \mathrm{Sp}(2r,\mathbb{C}) = \det\left[\binom{2(2r+1)-2i-2j}{(2r+1)-2i-1}\right]_{1\le i,j\le r}.$$

Proof Comparing the first two rows in Table 2, we see that the Weyl groups for $\mathrm{SO}(2r+1,\mathbb{C})$ and $\mathrm{Sp}(2r,\mathbb{C})$ have the same cardinality, the Coxeter exponents are equal, the convex hull of the weights are equal, and there is a natural bijection between the coroots. In fact, among the r^2 coroots for $\mathrm{SO}(2r+1,\mathbb{C})$ and $\mathrm{Sp}(2r,\mathbb{C})$, $r(r-1)$ are equal and r differ by a factor of 2 with the coroots for $\mathrm{Sp}(2r,\mathbb{C})$ being larger. Hence, Theorem 2.4 implies that $\deg \mathrm{SO}(2r+1,\mathbb{C}) = 2^{2r} \deg \mathrm{Sp}(2r,\mathbb{C})$ and Theorem 1.1 shows that $\deg \mathrm{Sp}(2r,\mathbb{C}) = \det\big[\binom{2(2r+1)-2i-2j}{(2r+1)-2i-1}\big]_{1\le i,j\le r}$. □

4 Non-intersecting Lattice Paths

This section gives a combinatorial interpretation for the determinant appearing in our formulas for the degree of $\mathrm{SO}(n,\mathbb{C})$. In particular, we show that this determinant counts appropriate collections of non-intersecting lattice paths by using the celebrated Lindström–Gessel–Viennot Lemma; see [1, Chap. 29] or [10, Theorem 1].

To sketch this approach, let Q be a locally-finite directed acyclic graph. Since there are no directed cycles in Q and every vertex in Q is the tail of only finitely many arrows, it follows that there are only finitely many directed paths (connected sequences of distinct arrows all oriented in the same direction) between any two vertices. For pair a, b of vertices in Q, let $m_{a,b} \in \mathbb{N}$ be number of directed paths from a to b. Given two finite lists $A := \{a_1, a_2, \ldots, a_r\}$ and $B := \{b_1, b_2, \ldots, b_r\}$ of vertices, the associated *path matrix* is $M := [m_{a_i,b_j}]_{1 \leq i,j \leq r} \in \mathbb{N}^{r \times r}$. A *path system P* from A to B consists of a permutation $\sigma \in \mathfrak{S}_r$ together with r directed paths from a_i to $b_{\sigma(i)}$. For $\sigma \in \mathfrak{S}_r$, set $\operatorname{sgn}(\sigma) := (-1)^k$ where k is the number of inversions in σ. If the paths in P are pairwise vertex-disjoint, then P is a *non-intersecting* path system. The following "lemma" relates $\det M$ with non-intersecting path systems.

Lemma 4.1 (Lindström–Gessel–Viennot) *If A and B are finite lists, having the same cardinality and consisting of vertices from a locally-finite directed acyclic graph, then the determinant of the associated path matrix M equals the signed sum of the non-intersecting path systems from A to B:* $\det M = \sum_P \operatorname{sgn}(\sigma)$. □

For our application, consider the directed grid graph whose vertices are the lattice points in $\mathbb{Z}^2$ and whose arrows are unit steps in either the north or east direction. In other words, the vertex $(i, j) \in \mathbb{Z}^2$ is the tail of exactly two arrows: one with head $(i, j+1)$ and the other with head $(i+1, j)$. The next result provides our combinatorial reinterpretation for the degree of $\mathrm{SO}(n, \mathbb{C})$.

Proposition 4.2 *Let $n \in \mathbb{N}$. If $N(n)$ is the number of non-intersecting path systems in the directed grid graph from $A := \{(2-n, 0), (4-n, 0), \ldots, (2\lfloor n/2 \rfloor - n, 0)\}$ to $B := \{(0, n-2), (0, n-4), \ldots, (0, n - 2\lfloor n/2 \rfloor)\}$, then we have*

$$\deg \mathrm{SO}(n, \mathbb{C}) = 2^{n-1} N(n)\,.$$

Proof By construction, the only non-intersecting path systems in our directed grid graph have direct paths from $(2i - n, 0)$ to $(0, n - 2i)$ for $0 \leq i \leq \lfloor n/2 \rfloor$. Hence, the associated element in $\mathfrak{S}_{\lceil n/2 \rceil}$ is the identity permutation and the determinant of the associated path matrix counts the total number of non-intersecting path systems.

The number of directed paths from $(0, 0)$ to (i, j) in our directed grid graph is $\binom{i+j}{i}$; simply choose which i arrows in the connected sequence are oriented east. Since the grid graph is invariant under translation, it follows that the number of direct paths from the vertex $(2i - n, 0)$ to $(0, n - 2j)$ equals $\binom{2n-2i-2j}{n-2i}$. Therefore, the path matrix associated to A and B is $M = \left[\binom{2n-2i-2j}{n-2i}\right]_{1 \leq i,j \leq \lfloor n/2 \rfloor}$. Combining Theorem 1.1 and Lemma 4.1, we conclude that $\deg \mathrm{SO}(n, \mathbb{C}) = 2^{n-1} N(n)$. □

Remark 4.3 From Corollaries 3.3–3.4, we also see that $\deg \mathrm{O}(n, \mathbb{C}) = 2^n N(n)$ and $\deg \mathrm{Sp}(2r, \mathbb{C}) = N(2r + 1)$.

Example 4.4 For $n = 5$, the 24 non-intersecting path systems are illustrated in Fig. 1. It follows that $\deg \mathrm{SO}(5, \mathbb{C}) = 2^4(24) = 384$.

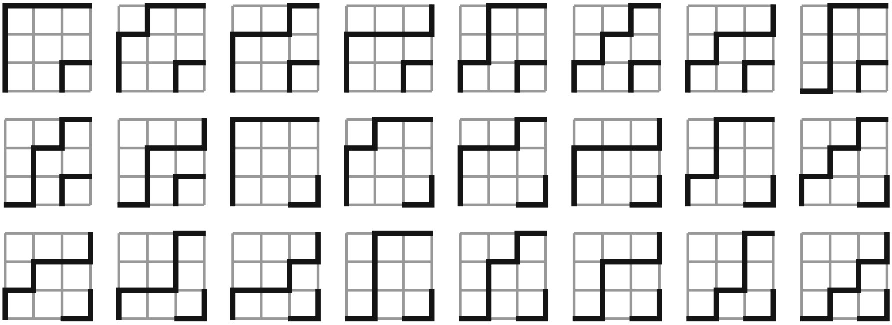

Fig. 1 The non-intersecting path systems from $\{(-3,0),(-1,0)\}$ to $\{(0,1),(0,3)\}$

Theorem 4.2 suggests that there might be a deeper relationship between the degree of SO($n, \mathbb{C}$) and lattice paths. It would be interesting to find a direct connection. Since the degree of Sp($2r, \mathbb{C}$) does not have a coefficient involving a power of 2, it may be the natural place to look for a combinatorial proof.

5 The Degree of a Low-Rank Optimization Problem

In this section, we show how the degree of SO($n, \mathbb{C}$) arises in counting the number of critical points for a particular optimization problem.

To motivate our particular problem, we first consider a more general framework. The *trace* tr(A) of a square matrix C is the sum of the entries on the main diagonal, and a real symmetric matrix X is *positive semidefinite*, written $X \succeq 0$, if all of its eigenvalues are nonnegative. A semidefinite programming problem has the form:

For real symmetric matrices $C, A_1, A_2, \ldots, A_m \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^m$, minimize $\operatorname{tr}(CX)$, for all real symmetric matrices $X \in \mathbb{R}^{n\times n}$, subject to the constraints that $X \succeq 0$ and $\operatorname{tr}(A_iX) = b_i$ for all $1 \le i \le m$. (SDP)

Many practical problems can be modeled as, and many NP-hard problems can be approximated by, semidefinite programming problems; see [3, 11]. Although semidefinite programming problems can often be efficiently solved by interior point methods, this invariably becomes computationally prohibitive for large n. Since the rank of an optimal solution is often much smaller than n, Burer and Monteiro [5] study the hierarchy of relaxations in which X is replaced by the low-rank positive semidefinite matrix RR^T. Specifically, the new optimization problem is:

For real symmetric matrices $C, A_1, A_2, \ldots, A_m \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^m$, minimize $\operatorname{tr}(CRR^\mathsf{T})$, for all $R \in \mathbb{R}^{n\times r}$, subject to the constraints that $\operatorname{tr}(A_iRR^\mathsf{T}) = b_i$ for all $1 \le i \le m$. (NOP)

When $r < (n+1)/2$, this alternative formulation has the advantage of reducing the number of unknowns from $\binom{n+1}{2}$ to nr. However, the objective function and the contraints are no longer linear—they are quadratic and the feasible set is non-convex.

Burer and Monteiro [5] propose a fast algorithm for solving (NOP). Despite the existence of multiple local minima, this algorithm quickly finds the global minimum in practice. To help understand this phenomenon, we examine the *critical points*, those points where the partial derivatives of the associated Lagrangian function vanish, of (NOP). Before giving our formula for the number of critical points of the new optimization problem, we need the following notation.

Definition 5.1 For positive integers i and j, let $\psi_i := 2^{i-1}$, let $\psi_{0,j} := \psi_j$, and let $\psi_{i,j} := \sum_{k=i}^{j-1} \binom{i+j-2}{k}$. For $r > 2$, set

$$\psi_{i_1,i_2,\ldots,i_r} := \begin{cases} \operatorname{pf}\left[\psi_{i_k,i_\ell}\right]_{1\le k<\ell\le r} & \text{if } r \text{ is even} \\ \operatorname{pf}\left[\psi_{i_k,i_\ell}\right]_{0\le k<\ell\le r} & \text{if } r \text{ is odd,} \end{cases}$$

where pf denotes the Pfaffian of a skew-symmetric matrix. For positive integer m and n, we define $\delta(m,n,r) := \sum_I \psi_I \psi_{I'}$, where the sum runs over all strictly increasing subsequences $I := \{i_1, i_2, \ldots, i_{n-r}\}$ of $\{1, 2, \ldots, n\}$ with $i_1 + i_2 + \cdots + i_{n-r} = m$ and $I' := \{1, 2, \ldots, n\} \setminus I$ denotes the complement.

Remark 5.2 Originally defined in [16] as the number of critical points for (SDP) in which the matrix X has rank r, the number $\delta(m,n,r)$ is called the *algebraic degree* of the semidefinite programming problem. Our defining formula for $\delta(m,n,r)$ was subsequently computed in [2].

Theorem 5.3 *The number of critical points for* (NOP) *is* $2\delta(m,n,r) \deg \mathrm{SO}(r,\mathbb{C})$.

Proof Given new variables $y_1, y_2, \ldots, y_m$, the Lagrangian function associated to (NOP) is $L(R,y) := \operatorname{tr}(CRR^\mathsf{T}) - \sum_{i=1}^m y_i\big(\operatorname{tr}(A_iRR^\mathsf{T}) - b_i\big)$. Taking the partial derivatives of $L(R,y)$ yields the equations

$$\left(C - \sum_{i=1}^m y_iA_i\right) RR^\mathsf{T} = 0 \qquad \text{and} \qquad \operatorname{tr}(A_iRR^\mathsf{T}) = b_i\,, \quad \text{for } 1 \le i \le m,$$

which define the set of critical points. Analogously, the critical points for (SDP) are determined by the equations

$$\left(C - \sum_{i=1}^m y_iA_i\right) X = 0 \qquad \text{and} \qquad \operatorname{tr}(A_iX) = b_i\,, \quad \text{for } 1 \le i \le m.$$

Nie, Ranestad, and Sturmfels [16] show that the number of critical points for (SDP), for which the rank of X equals r, is $\delta(m,n,r)$. Comparing the defining systems of equations for the critical points of (NOP) and (SDP), we see that the fibre of the map $(R,y) \mapsto (RR^\mathsf{T}, y)$ over each point (X,y) consists of all points (R,y') for which

$X = RR^{\mathsf{T}}$ and $y' = y$. Given X and R such that $X = RR^{\mathsf{T}}$, all other matrices S such that (S, y) lies in the fibre over (X, y) have the form $S = RU$ where U is an orthogonal $(r \times r)$-matrix. In other words, the fibre is isomorphic to a copy of the orthogonal group. Therefore, the number of critical points for (NOP) equals $2\delta(m, n, r) \deg \mathrm{SO}(r, \mathbb{C})$. □

Since the number of critical points for (NOP) grows rapidly with the rank r, the appealing behaviour of the algorithm in [5] still needs to be explained.

Remark 5.4 For applications, the most important critical points for (NOP) are real and satisfy the equation $\left(C - \sum_{i=1}^{m} y_i A_i\right) \succeq 0$.

6 Computational Methods

Since Theorem 1.1 provides a formula for the degree of $\mathrm{SO}(n, \mathbb{C})$, this family of examples becomes an interesting testing ground for various symbolic and numerical methods for computing degrees. In this section, we outline three algorithmic techniques for calculating the degree of a variety. The first is based on Gröbner bases, the second uses polynomial homotopy continuation, and the third involves numerical monodromy. Table 1 summarizes the results of our computations, and the related *Macaulay2* code appears in the Appendix. Beyond contrasting these algorithms, we hope that the different routines and auxiliary data, such as Gröbner bases or witness sets, will lead to new insights into the degrees of varieties.

The standard symbolic algorithm for determining the degree of a variety first finds a Gröbner basis of the defining ideal and then uses combinatorial properties of the initial ideal to return the Hilbert polynomial; the degree can be easily extracted from the highest degree term of the Hilbert polynomial. As this method is independent of the ground field, one can speed up the calculation by working over a small finite field. With this algorithm, we were able to compute the degree of $\mathrm{SO}(n, \mathbb{C})$ for all $2 \leq n \leq 5$, but it was the slowest among the methods we compared.

The basic numerical strategy for computing the degree of $\mathrm{SO}(n, \mathbb{C})$ randomly chooses a linear subspace L of complementary dimension and counts the number of complex solutions S to the zero-dimensional system of polynomial equations corresponding to $\mathrm{SO}(n, \mathbb{C}) \cap L$. The triple $\big(\mathrm{SO}(n, \mathbb{C}), L, S\big)$ is called a *witness set* for $\mathrm{SO}(n, \mathbb{C})$. This triple is a fundamental data type in numerical algebraic geometry: the computation of a witness set is often a necessary input to other numerical algorithms, including sampling points on the variety, studying its asymptotic behaviour, computing its monodromy group, or even studying its real locus; see Sect. 7. Both numerical algorithms presented below produce a witness set for $\mathrm{SO}(n, \mathbb{C})$.

Polynomial homotopy continuation computes a witness set by finding numerical approximations for the complex solutions S. First, one constructs a polynomial system that has a similar structure to the target system and has a simple solution set. This start system is embedded in a homotopy relating it to the target system and the numerical solutions of the start system are traced towards solutions of the target system. Start systems correspond to root counts. For dense systems, one typically uses the Bézout bound whereas, for sparse systems, one uses the mixed volume of the appropriate Newton polytopes. However, for $\mathrm{SO}(n, \mathbb{C})$, both of these bounds are equal $2^{n(n+1)/2}$, which grows quickly (for $n = 6$, it is already 2 097 152). Because of the number of paths that must be tracked, we were only able to compute the degree of $\mathrm{SO}(n, \mathbb{C})$ for all $2 \leq n \leq 5$ using this method.

Our third technique takes advantage of monodromy; see [7]. Suppose L and L' are two linear subspaces of complementary dimension to $\mathrm{SO}(n, \mathbb{C})$. Given a point on the linear slice $W := \mathrm{SO}(n, \mathbb{C}) \cap L$, we can numerically track this solution along some path γ to a point in another slice $W' := \mathrm{SO}(n, \mathbb{C}) \cap L'$. Tracking the second point along a different path γ' back to W yields another point in W and induces a permutation $\sigma_{\gamma,\gamma'}$ on the points in W. Iterating this process, one expects to populate the witness set associated to W. Although there are algorithms [17] which certify that a witness set is complete, one frequently uses heuristic stopping criteria because they are much faster. This monodromy method is implemented in the *MonodromySolver* package for *Macaulay2* [9]. With the naive stopping criterion that no new points were found after ten consecutive iterations, we were able to calculate with this method the degree $\mathrm{SO}(n, \mathbb{C})$ for all $6 \leq n \leq 7$.

7 Real Points on $\mathrm{SO}(n, \mathbb{C})$

Motivated by the applications to optimization, this section investigates the structure of the real points in $\mathrm{SO}(n, \mathbb{C})$. Taking advantage of the numerical monodromy algorithm, we collect experimental data counting the number of real points in witness sets for $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SO}(4, \mathbb{C})$, and $\mathrm{SO}(5, \mathbb{C})$.

More precisely, we use the `random` function in *Macaulay2* [9] to generate a sample of linear slices of $\mathrm{SO}(n, \mathbb{C})$. Homotopy continuation allows us to track solutions from a precomputed witness set to those lying on each randomly chosen linear slice. We determine how many solutions in the random slice are real by checking whether each coordinate is within a 0.001 numerical tolerance of being real. One can actually certify reality using *alphaCertify* [12], which implements Smale's α-theory. However, for the sake of speed, we limited these formal checks to at least one witness set achieving the maximum observed number of real points. The results of computing 1,398,000, 1,004,100, and 48,200 witness sets for $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SO}(4, \mathbb{C})$, and $\mathrm{SO}(5, \mathbb{C})$ are displayed in Figs. 2 and 3.

The raw data and actually code can be found at [4]. In rare examples, the process failed to return a witness set on the randomly chosen linear slice, because the homotopy continuation was ill-conditioned. In particular, we observed 2, 51, and 81 such failures for $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SO}(4, \mathbb{C})$, and $\mathrm{SO}(5, \mathbb{C})$ respectively. Despite the fact that all witness sets computed for $\mathrm{SO}(4, \mathbb{C})$ and $\mathrm{SO}(5, \mathbb{C})$ had fewer than 40 and 384 solutions, we are not convinced that there exists a non-trivial upper bound for the number of real solutions on a witness set of $\mathrm{SO}(n, \mathbb{C})$ exists. In fact, we conjecture that, for all $n \geq 2$, $\mathrm{SO}(n, \mathbb{C})$ admits a real witness set.

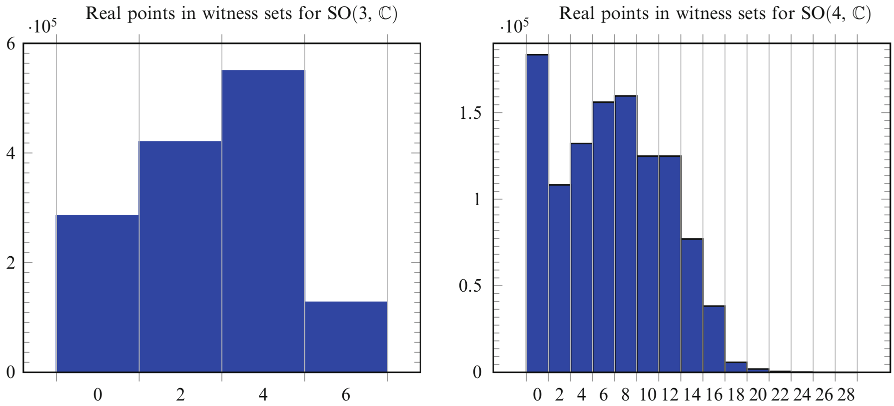

Fig. 2 Some histograms for the number of real solutions found in each witness set

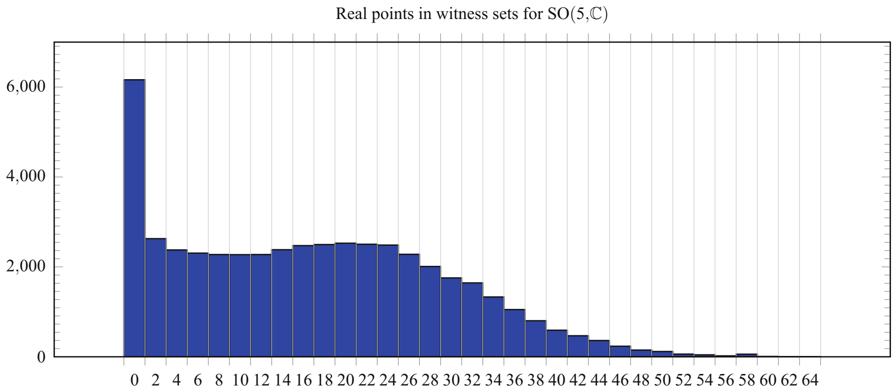

Fig. 3 Another histogram for the number of real solutions found in each witness set

Acknowledgements This article was initiated during the Apprenticeship Weeks (22 August–2 September 2016), led by Bernd Sturmfels, as part of the Combinatorial Algebraic Geometry Semester at the Fields Institute. The authors are very grateful to Jan Draisma for his tremendous help with understanding the Kazarnovskij Formula and to Kristian Ranestad for many helpful discussions. The authors thank Anton Leykin for performing the computation of SO($7,\mathbb{C}$). The first three authors would also like to thank the Max Planck Institute for Mathematics in the Sciences in Leipzig, Germany for their hospitality where some of this article was completed. The motivation for computing the degree of the orthogonal group came from project that started by the fifth author at the suggestion of Benjamin Recht. The first author was supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE 1106400, and the second author was partially supported by the NSF GRFP under Grant No. DGE-1256259 and the Wisconsin Alumni Research Foundation.

Appendix: *Macaulay2* Code

This section contains *Macaulay2* [9] code for computing the degree of $SO(n, \mathbb{C})$. We typically compute the degree of $O(n, \mathbb{C})$, and divide by to 2 to obtain the degree of $SO(n, \mathbb{C})$, because this approach eliminates the polynomial of highest degree, the condition that the determinant equal 1.

First, we compute the degree of SO(5) using Gröbner bases. The computation is done over the finite field $\mathbb{Z}/2\mathbb{Z}$ for $O(5, \mathbb{C})$ and the result is halved to give the degree of $SO(5, \mathbb{C})$.

```
deg1SO = n -> (
    R := ZZ/2[x_(1,1)..x_(n,n)];
    M := genericMatrix(R,n,n);
    J := minors(1, M * transpose(M) - id_(R^n));
    (degree J) // 2)
```

Our second function uses the package *NumericalAlgebraicGeometry* to solve the zero-dimensional system arising from a linear slice of the variety $O(3, \mathbb{C})$. The command `solveSystem` employs the standard method of polynomial homotopy continuation.

```
needsPackage ''NumericalAlgebraicGeometry'';
deg2SO = n -> (
    R := CC[x_(1,1)..x_(n,n)];
    M := genericMatrix(R,n,n);
    B := M * transpose(M) - id_(R^n);
    polys := unique flatten entries B;
    linearSlice := apply(binomial(n,2),
        i -> random(1,R) - random(CC));
    S := solveSystem(polys | linearSlice);
    #S // 2)
```

We next provide code that computes the degree of $SO(n, \mathbb{C})$ using the package *MonodromySolver*. Again we do not include the determinant condition, but this time we do *not* need to halve the result. This is because our starting point, the identity matrix, lies on $SO(n, \mathbb{C})$ and this method only discovers points on the irreducible component corresponding to our starting point. The linear slices are parametrized by the t and c variables which are varied within the function `monodromySolve` to create monodromy loops. The method stops when ten consecutive loops provide no new points. Although it is possible that this stopping criterion is satisfied prematurely, in our case the program stopped at the correct number.

```
needsPackage ''MonodromySolver'';
deg3SO = n -> (
    d := binomial(n,2);
    R := CC[c_1..c_d,
        t_(1,1,1)..t_(d,n,n)][x_(1,1)..x_(n,n)];
    M := genericMatrix(R,n,n);
```

```
    B := M * transpose(M) - id_(R^n);
    polys := unique flatten entries B;
    linearSlice := for i from 1 to d list (
        c_i + sum flatten for j from 1 to n list (
            for k from 1 to N list t_(i,j,k)*x_(j,k)));
    G := polySystem( polys | linearSlice);
    setRandomSeed 0;
    (p0, x0) := createSeedPair(G,
        flatten entries id_(CC^n));
    (V, npaths) = monodromySolve(G, p0, {x_0},
        NumberOfNodes => 2, NumberOfEdges => 4);
    # flatten points V.PartialSols)
```

Finally, we may use Theorem 1.1 to compute the degree of SO($n, \mathbb{C}$).

```
deg4SO = n -> (
    r := n // 2;
    M := matrix table(toList(1..r), toList(1..r),
        (i,j) -> binomial(2*n-2*i-2*j, n-2*i));
    2^(n-1) * det(M))
```

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