Multigraded Castelnuovo–Mumford Regularity on Products of Projective Space

Juliette Bruce
(joint work with Lauren Cranton Heller, Mahrud Sayrafi)

1. Castelnuovo–Mumford regularity on Projective Spaces

Before discussing multigraded Castelnuovo–Mumford regularity on products of projective spaces and our new work, we begin by briefly recalling the standard graded story of Castelnuovo–Mumford regularity on a single projective space. Introduced by Mumford in the mid-1960's Castelnuovo–Mumford regularity is defined in terms of cohomological vanishing.

Definition 1.1. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is d-regular if and only if:

$$H^i(\mathbb{P}^n, \mathcal{F}(d-i)) = 0 \qquad \text{for all } i > 0.$$

The Castelnuovo–Mumford regularity of $\mathcal{F}$ is then

$$\operatorname{reg}(\mathcal{F}) := \min\left\{d \in \mathbb{Z} \;\middle|\; \mathcal{F} \text{ is } d\text{-regular}\right\}.$$

Roughly speaking one should think about Castelnuovo–Mumford regularity as being a measure of geometric complexity. Mumford was interested in such a measure as it plays a key role in constructing Hilbert and Quot schemes. In particular, being d-regular implies that $\mathcal{F}(d)$ is globally generated. However, in the 1980's Eisenbud and Goto showed that being d-regular was also closely connected to interesting homological properties.

Theorem 1.2. [4] *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and $M = \bigoplus_{e\in\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(e))$ the corresponding section ring. The following are equivalent:*

- *M is d-regular;*
- *$\beta_{i,j}(M) := \dim_{\mathbb{K}} \operatorname{Tor}_i(M, \mathbb{K})_j = 0$ for all $i \geq 0$ and $j > d + i$;*
- *$M_{\geq d}$ has a linear resolution.*

The goal of our work is to try and understand how this theorem may be generalized to the multigraded setting, i.e. from coherent sheaves on a single projective space to sheaves on a product of projective spaces.

2. Multigraded Setting: Products of Projective Spaces

Shifting to the multgraded setting, we fix a dimension vector $\mathbf{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ and let $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$. We then let $S = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq r, 0 \leq j \leq n_i]$ be the Cox ring of $\mathbb{P}^{\mathbf{n}}$ with the $\operatorname{Pic}(X) \cong \mathbb{Z}^r$-grading given by $\deg x_{i,j} = \mathbf{e}_i \in \mathbb{Z}^r$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{Z}^r$.

Maclagan and Smith generalized Castelnuovo–Mumford regularity to this setting in terms of certain cohomology vanishing. Before we can state their definition of multigraded regularity we need to fix some useful notation to described the regions in which we will require cohomology to vanish.

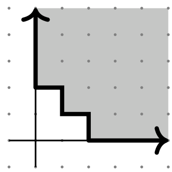
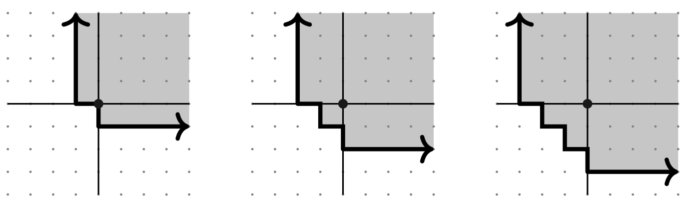


FIGURE 1. The multigraded Castelnuovo–Mumford regularity of $\mathcal{O}_X$ where $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the subscheme consisting of three distinct points $([1:1],[1:4])$, $([1:2],[1:5])$, and $([1:3],[1:6])$.

Notation 2.1. Given $\mathbf{d} \in \mathbb{Z}^r$ and $i \in \mathbb{Z}_{\geq 0}$ we let:

$$L_i(\mathbf{d}) \coloneqq \bigcup_{\substack{\mathbf{v}\in\mathbb{N} \\ |\mathbf{v}|=i}} (\mathbf{d}-\mathbf{v}) + \mathbb{N}^r.$$

In order to get a sense for what these regions look like note when $r = 2$ the region $L_i(\mathbf{d})$ looks like a staircase with $(i+1)$-corners. Below we've plotted the regions $L_1(0,0)$, $L_2(0,0)$, and $L_3(0,0)$. Roughly speaking we are going to define regularity by require H^i to vanish on L_i.



With this notation in hand we recall the notion of multigraded Castelnuovo–Mumford regularity as introduced by Maclagan and Smith.

Definition 2.2. [5, Definition 6.1] A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{\mathbf{n}}$ is $\mathbf{d}$-regular if and only if

$$H^i(\mathbb{P}^{\mathbf{n}}, \mathcal{F}(\mathbf{e})) = 0 \qquad \text{for all } \mathbf{e} \in L_i(\mathbf{d}).$$

The multigraded Castelnuovo–Mumford regularity of $\mathcal{F}$ is then the set:

$$\operatorname{reg}(\mathcal{F}) \coloneqq \left\{\mathbf{d} \in \mathbb{Z}^r \;\middle|\; \mathcal{F} \text{ is } \mathbf{d}\text{-regular}\right\} \subset \mathbb{Z}^r.$$

Even for relatively simple examples the multigraded Castelnuovo–Mumford regularity does not necessarily have a unique minimal element (see Figure 2). That said $\operatorname{reg}(\mathcal{F})$ does have the structure of a module over the semi-group $\operatorname{Nef}(\mathbb{P}^{\mathbf{n}}) \cong \mathbb{N}^r$, i.e. if $\mathbf{d} \in \operatorname{reg}(\mathcal{F})$ then $\mathbf{d} + \mathbf{e} \in \operatorname{reg}(\mathcal{F})$ for all $\mathbf{e} \in \mathbb{N}^r$.

The obvious approaches to generalize Theorem 1.2 to a product of projective spaces turn out not to work. For example, the multigraded Betti numbers do not determine multigraded Castelnuovo–Mumford regularity [2, Example 5.1] With this in mind we focus on generalizing part (3) of Theorem 1.2.

Definition 2.3. [2] Let $F_\bullet$ be a complex of $\mathbb{Z}^r$-graded free S-modules.

1. (1) We say that $F_\bullet$ is $\mathbf{d}$-linear if and only if F_0 is generated in degree $\mathbf{d}$ and each twist of F_i is contained in $L_i(\mathbf{d})$.
2. (2) We say that $F_\bullet$ is $\mathbf{d}$-quasilinear if and only if F_0 is generated in degree $\mathbf{d}$ and each twist of F_i is contained in $L_{i-1}(\mathbf{d}-\mathbf{1})$.

In order to see the difference between linear and quasilinear resolutions we note that on a product of projective spaces the irrelevant ideal generally will have a quasilinear resolution, not a linear resolution. For example, if we consider $\mathbb{P}^1 \times \mathbb{P}^2$ so that $S = \mathbb{K}[x_0, x_1, y_0, y_1, y_2]$ and $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle$ then the minimal graded free resolution of S/B is:

$$S \longleftarrow S(-1,-1)^6 \longleftarrow \begin{matrix} S(-1,-2)^6 \\ \oplus \\ S(-2,-1)^3 \end{matrix} \longleftarrow \begin{matrix} S(-1,-3)^2 \\ \oplus \\ S(-2,-2)^3 \end{matrix} \longleftarrow S(-2,-3) \longleftarrow 0.$$

In particular, we see that the minimal graded free resolution S/B is not $(0,0)$-linear since $(-1,-1) \notin L_1(0,0)$, however, it is $(0,0)$-quasilinear.

It is not the case that M being $\mathbf{d}$-regular implies $M_{\geq \mathbf{d}}$ has a linear resolution [2, Example 4.2], however, we can characterize being $\mathbf{d}$-regular in terms of $M_{\geq \mathbf{d}}$ having a quasilinear resolution.

Theorem 2.4. [2, Theorem A] *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module with $H^0_B(M) = 0$ then:*

$$M \textit{ is } \mathbf{d}\textit{-regular} \iff M_{\geq \mathbf{d}} \textit{ has a } \mathbf{d}\textit{-quasilinear resolution}$$

We briefly sketching the proof of the above theorem:

1. (1) Using a Fourier-Mukai argument we construct a complex $G_\bullet$ of free $\mathbb{Z}^r$-graded S-modules whose multigraded Betti numbers are given (in some range) as follows:
$$\beta_{i,\mathbf{a}}(G_\bullet) = \dim H^{|\mathbf{a}|-i}\left(\mathbb{P}^{\mathbf{n}}, \tilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\right).$$
2. (2) Making use of a spectral sequence argument we show that even though $G_\bullet$ is not a priori a resolution of $M_{\geq \mathbf{d}}$ we have that:
$$\beta_{i,\mathbf{a}}(M_{\geq \mathbf{d}}) = \beta_{i,\mathbf{a}}(G_\bullet).$$
3. (3) Finally, we characterize M being $\mathbf{d}$-regular in terms of the vanishing of the cohomology in (1) above.

Note the complex $G_\bullet$ constructed in part (1) of the proof sketch above is a priori not a resolution of $M_{\geq \mathbf{d}}$, but instead is a virtual resolution of M [1]. That said as noted above it does have the same Betti numbers as $M_{\geq \mathbf{d}}$, and in all the examples we have done it turns out to be a resolution.

Conjecture 2.5. [2, Conjecture 6.7] *The complex $G_\bullet$ is the minimal free resolution of $M_{\geq \mathbf{d}}$.*

3. Further Questions

Since computing the minimal graded free resolution of $M_{\geq \mathbf{d}}$ can be effectively done via Gröbner basis methods, Theorem 2.4 provides an efficient algorithm for checking whether a module is $\mathbf{d}$-regular for a particular $\mathbf{d} \in \mathbb{Z}^r$. It would be interesting to know whether such an algorithm could be extended to computing all of the minimal elements of $\operatorname{reg}(M)$.

Question 3.1. Is there an effective algorithm for computing the multigraded Castelnuovo–Mumford regularity of a coherent sheaf or module on $\mathbb{P}^{\mathbf{n}}$?

This is equivalent to finding a finite box in $\mathbb{Z}^r$ that contains all of the minimal elements of $\operatorname{reg}(M)$. If such a finite box does exist, it is very special to the case of a product of projective spaces.

In particular, one may consider multigraded Castelnuovo–Mumford regularity of sheaves and modules on other toric varieties [5]. It turns out that there are examples of finitely generated modules on Hirzebruch surfaces whose multigraded Castelnuovo–Mumford regularity does not lie in finite a box [3]. This naturally leads one to ask what assumptions one needs to avoid such potential issues.

Question 3.2. Let X be a smooth projective toric variety, and M a finitely generated $\operatorname{Pic}(X)$-graded $\operatorname{Cox}(X)$-module. Under what assumptions is $\operatorname{reg}(M)$ finitely generated as a module over the semi-group $\operatorname{Nef}(X)$?

References

[1] Christine Berkesch, Daniel Erman, and Gregory G. Smith, *Virtual resolutions for a product of projective spaces*, Algebr. Geom. **7** (2020), no. 4, 460–481.

[2] Juliette Bruce, Lauren Cranton Heller, and Mahrud Sayrafi, *Characterizing Multigraded Regularity on Products of Projective Spaces* (2021). ArXiv pre-print: `https://arxiv.org/abs/2110.10705`.

[3] ———, *Bounds on the Multigraded Regularity of Modules and Powers of Ideals* (2022). To appear.

[4] David Eisenbud and Shiro Goto, *Linear free resolutions and minimal multiplicity*, Journal of Algebra **88** (1984), no. 1, 89–133.

[5] Diane Maclagan and Gregory G. Smith, *Multigraded Castelnuovo–Mumford regularity*, J. Reine Angew. Math. **571** (2004), 179–212.