Hilbert Functions in Algebra and Geometry

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Outline

What is a Hilbert function?

Hilbert's Theorem

Classification of Hilbert Functions in Geometry

Open questions

Graded rings

Definition

A commutative unital ring R is called a **graded ring** if it can be written as a direct sum of subgroups

$$R = \bigoplus_{i \geq 0} R_i \quad \text{such that} \quad R_i R_j \subseteq R_{i+j},\ \forall i, j \geq 0.$$

Elements of R_i are called *homogeneous elements* of degree i.

Example

- **polynomial rings** in several variables $R = \mathbb{F}[x_1, \ldots, x_n]$, R_i is the set of all homogeneous polynomials of degree i.
- the blowup (Rees) algebra $\mathcal{R}(I) = \bigoplus_{i \geq 0} I^i$ of any ideal I.

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If R is a graded ring and I is a homogeneous ideal then the ideal I as well as the quotient ring R/I are graded R-modules.

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Hilbert Function

From now

- $R = \mathbb{F}[x_1, \ldots, x_n]$
- M a finitely generated graded R-module.

Definition

The **Hilbert function** of a graded R-module M is given by

$$H_M : \mathbb{N} \to \mathbb{N}, \quad H_M(i) = \dim_{\mathbb{F}}(M_i).$$

Example/Exercise (Polynomial ring)

For $M = R = \mathbb{F}[x_1, \ldots, x_n]$, we have $H_M(i) = \binom{n+i-1}{i}$.

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Hilbert Function Example

Example

$I = (x^3y, x^2y^4) \subseteq R = \mathbb{F}[x, y]$

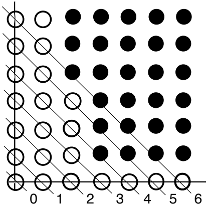

Figure: A picture of the ideal I

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$H_I(i)$	0	0	0	0	1	2	4	5	6	7	8	9	10
$H_{R/I}(i)$	1	2	3	4	4	4	3	3	3	3	3	3	3

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Patterns ?

- $H_I(i)$ grows linearly for $i \gg 0$: $H_I(i) = i - 2$ for $i \geq 6$.
- $H_{R/I}(i)$ eventually constant for $i \gg 0$: $H_{R/I}(i) = 3$ for $i \geq 6$.

Hilbert Series

Definition

The **Hilbert series** of a graded module M is the generating function

$$HS_M(t) = \sum_{i \geq 0} H_M(i) t^i.$$

Example (Polynomial ring)

For $M = R = \mathbb{F}[x_1, \ldots, x_n]$, we have $HS_M(t) = \frac{1}{(1-t)^n}$.

Hilbert Series Example

Example

If $R = \mathbb{F}[x_1, \ldots, x_n]$, then $HS_R(t) = \frac{1}{(1-t)^n}$.

Proof:

$$
\begin{aligned}
HS_R(t) &= \left(\frac{1}{1-t}\right)^n \Leftrightarrow \\
\sum_{i \geq 0} \dim_{\mathbb{F}}(R_i) t^i &= (1 + t + t^2 + \cdots t^a + \cdots)^n \Leftrightarrow \\
\dim_{\mathbb{F}}(R_i) &= \#\{(a_1, a_2, \ldots, a_n) \mid a_1 + a_2 + \cdots + a_n = i\} \Leftrightarrow \\
\dim_{\mathbb{F}}(R_i) &= \#\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in R_i\} \qquad \checkmark.
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Enter Hilbert

Figure: David Hilbert (1862-1943)

Hilbert-Serre Theorem

Theorem (Hilbert-Serre)

If M is a finitely generated graded module over the polynomial ring $R = F[x_1, \ldots, x_n]$ then

$$HS_M(t) = \frac{p(t)}{(1-t)^n} \text{ for some } p(t) \in \mathbb{Z}[t].$$

In reduced form one can write $HS_M(t) = \frac{h(t)}{(1-t)^d}$ for unique

- ***h*-polynomial** $h = h_0 + h_1 t + \ldots + h_s t^s \in \mathbb{Z}[t]$ with $h(1) \neq 0$; $h_0, h_1, \ldots, h_s$ is called the ***h*-vector** of M
- $d \in \mathbb{N}$, $0 \leq d \leq n$ called the **Krull dimension** of M.

Corollary (Hilbert)

The Hilbert function of M is eventually given by a polynomial function of degree equal to $d-1$ called the **Hilbert polynomial**.

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Properties of Hilbert Series

Proposition

1. **Additivity in short exact sequences:** *if* $0 \to A \to B \to C \to 0$ *is a short exact sequence of graded modules and maps then*

$$HS_B(t) = HS_A(t) + HS_C(t).$$

2. **Sensitivity to regular elements:** *if* M *is a graded module and* $f \in R_d$, $d \geq 1$, *is a non zero-divisor on* M *then*

$$HS_{M/fM}(t) = (1 - t^d)HS_M(t).$$

Hilbert Series Example

Example

For $R = \mathbb{F}[x, y, z]$ let's compute the Hilbert Series for

$$M = R/(\underbrace{x^2 + y^2 + z^2}_{f_1}, \underbrace{x^3 + y^3 + z^3}_{f_2}, \underbrace{x^4 + y^4 + z^4}_{f_3})$$

.

- f_1 is a non zero-divisor on R, thus $HS_{R/f_1)}(t) = (1 - t^2)HS_R(t)$
- f_2 is a non zero-divisor on $R/(f_1)$, thus

$$HS_{R/(f_1,f_2)}(t) = (1 - t^3)HS_{R/(f_1)}(t) = (1 - t^3)(1 - t^2)HS_R(t)$$

- f_3 is a non zero-divisor on $R/(f_1, f_2)$, thus

$$\begin{aligned} HS_{R/(f_1,f_2,f_3)}(t) &= (1 - t^4)HS_{R/(f_1,f_2)}(t) = (1 - t^4)(1 - t^3)(1 - t^2)HS_R(t) \\ &= \frac{(1 - t^4)(1 - t^3)(1 - t^2)}{(1 - t)^3} \\ &= t^6 + 3t^5 + 5t^4 + 6t^3 + 5t^2 + 3t + 1. \end{aligned}$$

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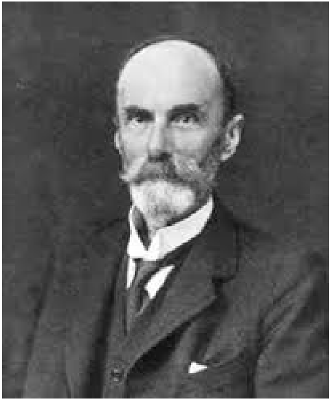
Classification of Hilbert functions



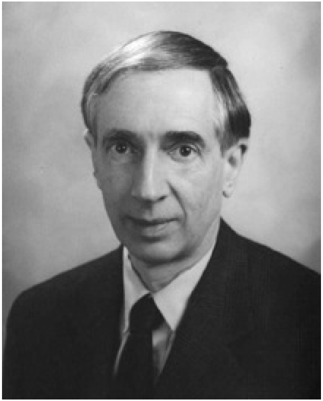


Figure: F. Macaulay (1862-1937) and R. Stanley.

Classification Problem

Question

What are all the possible Hilbert functions or Hilbert series or h-vectors of (cyclic) graded modules satisfying a given property?

Property of $M = R/I$	Description of H_M	Reference
Arbitrary	"admissible" (a combinatorial condition)	Macaulay
Complete intersection	$HS_M(t) = \frac{\prod_{i=1}^{s}(1-t^{d_i})}{(1-t)^n}$	the audience
Gorenstein	the h-vector must be symmetric	Stanley

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Geometric Classification Problem

Question

What are all the possible Hilbert functions of cyclic graded **domains** R/P*?*

- R/P is a domain iff P is a **prime** ideal
- the vanishing set of a prime ideal P,

$$V(P) = \{(a_1, \ldots, a_n) \in \mathbb{F}^n (\text{or } \mathbb{P}^{n-1}) \mid f(a_1, \ldots, a_n) = 0, \forall f \in P\}$$

is an irreducible **algebraic variety**

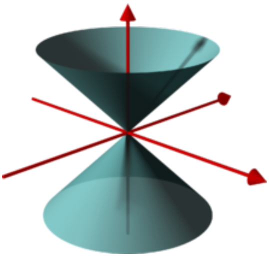

Figure: An algebraic variety $V(x^2 + y^2 - z^2)$.

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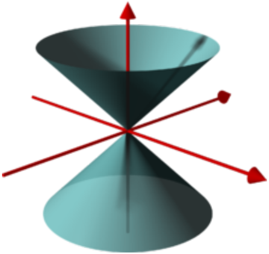

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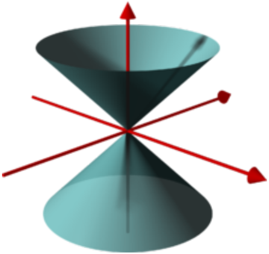

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Bertini's Theorem

Theorem (Bertini)

Let R/P be a Cohen-Macaulay[1] *domain of Krull dimension at least three over an infinite field $\mathbb{F}$. Then there exists $f \in R_1$ such that $R/P + (f)$ is also a domain.*

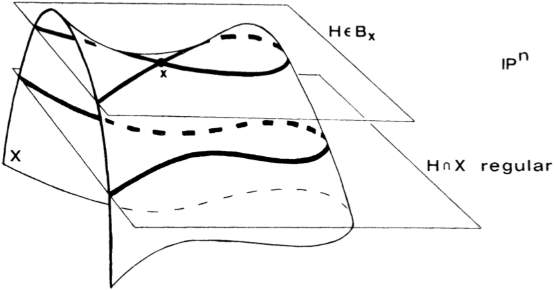


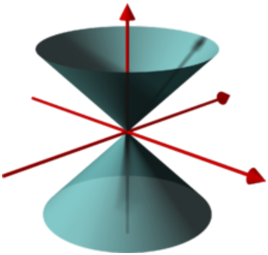
Figure: An illustration of Bertini's theorem.

[1] a technical condition which allows for induction on the Krull dimension.

Reduction to the case of curves

Corollary (Stanley)

Let R/P be a Cohen-Macaulay graded domain of dimension greater or equal than two. Then the h-vector of R/P is also the h-vector of a Cohen-Macaulay graded domain of Krull **dimension two** *(that is, the homogeneous coordinate ring of an irreducible projective curve).*



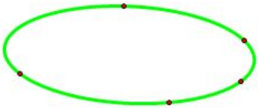


Figure: An algebraic variety $V(x^2 + y^2 - z^2)$ of Krull dimension two in affine space and in projective space.

Further reduction to points with UPP

Theorem (Harris)

*Let P be a prime ideal such that the Krull dimension of R/P is 2. Then there exists $f \in R_1$ (*a hyperplane*) such that $V(P + (f))$ (*the hyperplane section*) is a set Γ of d points such that for every subset $\Gamma' \subseteq \Gamma$ of d' points and for every $i \geq 0$ we have*

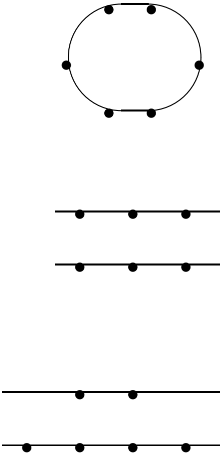
$$H_{I_\Gamma(i)} = \min\{d', H_{I'_\Gamma(i)}\}.$$

Definition

A set Γ of points satisfying the condition above is said to have the **uniform position property** (UPP).

UPP Example

Example/Exercise



h-vector 1 2 2 1 (complete intersection on a conic)

This has UPP.

h-vector 1 2 2 1 (complete intersection)

This has CB but not UPP.

h-vector 1 2 2 1

This has neither CB nor UPP.

Figure: Six points on a conic in $\mathbb{P}^2$ and the UPP.

Partial classification

Question (Reformulation of Classification Question)

What are all the possible Hilbert functions of points in $\mathbb{P}^n$ satisfying the uniform position property?

There is a partial answer in the case $n = 2$:

Theorem

A finite sequence of natural numbers is the h-vector of R/I, where $V(I)$ is a set of points in $\mathbb{P}^2$ satisfying UPP if and only if $h_0 = 1, h_1 = 2$ and the h-vector of R/I is **admissible** *and of* **decreasing type***, meaning if $h_{i+1} < h_i$ then $h_{j+1} < h_j$ for all $j \geq i$.*

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Open Problems

Figure: **You ?**

The Hilbert function of a generic algebra

Conjecture (Fröberg)

Let $F_1, \ldots, F_r$ be homogeneous polynomials of degrees $d_1, \ldots, d_r \geq 1$ in a polynomial ring $R = F[x_1, \ldots, x_n]$.
If $F_1, \ldots, F_r$ are chosen "randomly" and $I = (F_1, \ldots, F_r)$, then

$$HS_{R/I}(t) = \frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1 - t)^n}.$$

Stanley's unimodality conjecture

Conjecture (Stanley)

The h-vector of a graded Cohen-Macaulay domain is **unimodal**, *i.e. there exists* $0 \leq j \leq s$ *such that*

$$h_0 \leq h_1 \leq h_2 \ldots \leq h_j \geq \ldots \geq h_{s-1} \geq h_s.$$

Points with UPP

Question (Harris)

What are the possible Hilbert functions of points in $\mathbb{P}^n$, $n \geq 4$ satisfying the UPP?

Nagata's conjecture

An ideal defining a set of **fat points** is an ideal of the form

$$I = I_{p_1}^{m_1} \cap I_{p_2}^{m_2} \cap \cdots \cap I_{p_r}^{m_r}$$

where I_{p_i} is the ideal defining a point $p_i \in \mathbb{P}^n$.

Conjecture (Nagata)

If $I = I_{p_1}^{m_1} \cap I_{p_2}^{m_2} \cap \cdots \cap I_{p_r}^{m_r}$ is an ideal defining r fat points in $\mathbb{P}^n$ and $d > 0$ is an integer such that $H_I(d) > 0$ then

$$d \geq \frac{m_1 + m_2 + \cdots + m_r}{\sqrt{n}}.$$

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Thank you!