HILBERT FUNCTIONS IN ALGEBRA AND GEOMETRY (GWCAWMMG WORKSHOP)

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1. What is a Hilbert function?

Definition 1. A commutative unital ring R is called a *graded ring* if it can be written as a direct sum of subgroups

$$R = \bigoplus_{i \geq 0} R_i \quad \text{such that} \quad R_i R_j \subseteq R_{i+j},\ \forall i, j \geq 0.$$

Elements of R_i are called homogeneous elements of degree i.

Example.
- The main examples are polynomial ring in several variables $R = \mathbb{F}[x_1, \ldots, x_n]$, where R_i is the set of all homogeneous polynomials of degree i.
- For any ideal I, the blowup (Rees) algebra $\mathcal{R}(I) = \bigoplus_{i \geq 0} I^i$ is a graded ring with the given direct sum decomposition.

Definition 2. A module M over a graded ring R is called a *graded module* if it can be written as a direct sum of subgroups

$$M = \bigoplus_{j \geq 0} M_j \quad \text{such that} \quad R_i M_j \subseteq M_{i+j}\ \forall i, j \geq 0.$$

Example. If R is a graded ring and I is a homogeneous ideal (generated by homogeneous elements of R) then the ideal I as well as the quotient ring R/I are graded R-modules.

From now on we focus on $R = \mathbb{F}[x_1, \ldots, x_n]$ and M a finitely generated graded R-module. (The ideas presented here apply more generally when R is a graded ring with $R_0 = \mathbb{F}$ that is finitely generated as an $\mathbb{F}$-algebra.)

Definition 3. The *Hilbert function* of a graded module M is

$$H_M : \mathbb{N} \to \mathbb{N}, \quad H_M(i) = \dim_{\mathbb{F}}(M_i).$$

Example (Polynomial ring)**.** For $R = \mathbb{F}[x_1, \ldots, x_n]$, the Hilbert function is $H_R(i) = \binom{n+i-1}{i}$.

Example (Monomial ideal)**.** For $I = (x^3y, x^2y^4) \subseteq R = \mathbb{F}[x, y]$, the Hilbert function of I at i is the number of black dots on a diagonal with intercept $x = i$ while the Hilbert function of R/I at i is the number of white dots on a diagonal with intercept $x = i$.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$H_I(i)$	0	0	0	0	1	2	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$H_{R/I}(i)$	1	2	3	4	4	4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3

Patterns:
- $H_I(i)$ grows linearly for $i \gg 0$: $H_I(i) = i - 2$ for $i \geq 6$.
- $H_{R/I}(i)$ eventually constant for $i \gg 0$: $H_{R/I}(i) = 3$ for $i \geq 6$.

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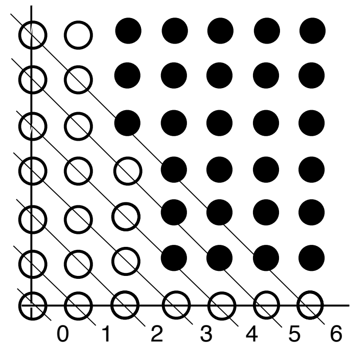

FIGURE 1. A picture of the ideal I

One often encodes a sequence of numbers into a generating series. The generating series for the Hilbert function is called the Hilbert series.

Definition 4. The *Hilbert series* of a graded module M is

$$HS_M(t) = \sum_{i \geq 0} H_M(i) t^i.$$

Example (Polynomial ring)**.** For $M = R = \mathbb{F}[x_1, \dots, x_n]$, we have $HS_M(t) = \frac{1}{(1-t)^n}$.

Proof for $n = 3$.

$$F[x, y, z] = \bigoplus_{i \geq 0} H_i, \quad H_i = \text{ homogeneous degree } i \text{ polynomials}$$

$$H_i = \mathrm{Span}_F \ \{x^a y^b z^c \mid a + b + c = i\}$$

$$\begin{aligned}\dim_F(H_i) &= [t^i] \Big((1 + t + \cdots + t^a + \cdots)(1 + t + \cdots + t^b + \cdots) \\ &\quad (1 + t + \cdots + t^c + \cdots)\Big) \\ &= [t^i] \left(\frac{1}{1-t} \cdot \frac{1}{1-t} \cdot \frac{1}{1-t} \right) = [t^i] \left(\frac{1}{(1-t)^3} \right)\end{aligned}$$

Thus

$$HS_{F[x,y,z]}(t) = \frac{1}{(1-t)^3}.$$

□

Example. For $R = \mathbb{F}[x, y, z]$ and

$$M = R/(x^2 + y^2 + z^2, x^3 + y^3 + z^3, x^4 + y^4 + z^4)$$

the Hilbert series is

$$HS_M(t) = \frac{(1-t^2)(1-t^3)(1-t^4)}{(1-t)^3} = t^6 + 3t^5 + 5t^4 + 6t^3 + 5t^2 + 3t + 1.$$

Patterns:

- the series HS_M can be written as a rational function with denominator $(1-t)^n$.
- in the second example, the nonzero coefficients form a symmetric and unimodal sequence.

2. Enter Hilbert

Theorem 5 (Hilbert-Serre)**.** *If M is a finitely generated graded module over the polynomial ring $R = F[x_1, \dots, x_n]$ then*

$$HS_M(t) = \frac{p(t)}{(1-t)^n} \text{ for some } p(t) \in \mathbb{Z}[t].$$

Considering the reduced form of the expression above, one can write $HS_M(t) = \frac{h(t)}{(1-t)^d}$ for *unique*

- $h = h_0 + h_1 t + \ldots + h_s t^s \in \mathbb{Z}[t]$ such that $h(1) \neq 0$; $h(t)$ is called the *h-polynomial* of M and $(h_0, h_1, \dots, h_s)$ is called the h-vector of M
- $d \in \mathbb{Z}$, $0 \leq d \leq n$ called the *Krull dimension* of M.

Corollary 6 (Hilbert)**.** *The Hilbert function of M is eventually given by a polynomial function of degree equal to $d-1$ called the Hilbert polynomial of M.*

The proof of this theorem involves graded free resolutions, which are beyond the scope of these notes. However the main properties involved in the proof are the following:

Proposition 7 (Properties of Hilbert Series)**.**

(1) *Additivity in short exact sequences: if $0 \to A \to B \to C \to 0$ is a short exact sequence of graded modules and maps then*

$$HS_B(t) = HS_A(t) + HS_C(t).$$

(2) *Sensitivity to regular elements: if M is a graded module and $f \in R_d$, $d \geq 1$ is a non zero-divisor on M then*

$$HS_{M/fM}(t) = (1-t^d) HS_M(t).$$

Example. For $R = \mathbb{F}[x, y, z]$ let's compute the Hilbert Series for

$$M = R/(\underbrace{x^2+y^2+z^2}_{f_1}, \underbrace{x^3+y^3+z^3}_{f_2}, \underbrace{x^4+y^4+z^4}_{f_3})$$

.

- f_1 is a non zero-divisor on R, thus $HS_{R/f_1)}(t) = (1-t^2) HS_R(t)$
- f_2 is a non zero-divisor on $R/(f_2)$, thus

$$HS_{R/(f_1,f_2)}(t) = (1-t^3) HS_{R/(f_1)}(t) = (1-t^3)(1-t^2) HS_R(t)$$

- f_3 is a non zero-divisor on $R/(f_1, f_2)$, thus

$$\begin{aligned} HS_{R/(f_1,f_2,f_3)}(t) &= (1-t^4) HS_{R/(f_1,f_2)}(t) = (1-t^4)(1-t^3)(1-t^2) HS_R(t) \\ &= \frac{(1-t^4)(1-t^3)(1-t^2)}{(1-t)^3} = t^6 + 3t^5 + 5t^4 + 6t^3 + 5t^2 + 3t + 1. \end{aligned}$$

Note that here every time we add one generator we also reduce the Krull dimension by one

Ring	R	$R/(f_1)$	$R/(f_1, f_2)$	$R/(f_1, f_2, f_3)$
Krull dimension	3	2	1	0

.

This property of $R/(f_1, f_2, f_3)$ is called being a *complete intersection*.

3. Classification of Hilbert Functions

Question 8. *What are all the possible Hilbert functions/ Hilbert series of graded modules $M = R/I$ satisfying a given property?*

Property of $M = R/I$	Description of H_M	Reference
Arbitrary	combinatorial condition	Macaulay [3]
Complete intersection	$HS_M(t) = \frac{\prod_{i=1}^{s}(1-t^{d_i})}{(1-t)^n}$	you, the audience
Gorenstein	the h-vector must be symmetric	Stanley [6]

For the rest of the notes we focus on the question

Question 9. *What are all the possible Hilbert functions of graded* **domains** *R/P?*

Recall that

- R/P is a domain iff P is a **prime** ideal
- the vanishing set of a (prime) ideal $V(P) = \{(a_1, \ldots, a_n) \in \mathbb{F}^n \mid f(a_1, \ldots, a_n) = 0, \forall f \in P\}$ is an (irreducible) **algebraic variety**

Then

- $H_P(d)$ is the number of linearly hypersurfaces of degree d that contain the variety $V(P)$.

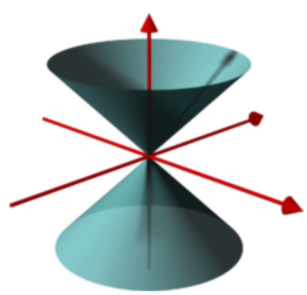

Figure 2. An algebraic variety $V(x^2 + y^2 - z^2)$ of Krull dimension two.

Theorem 10 (Bertini)**.** *Let R/P be a Cohen-Macaulay* [1] *domain of Krull dimension at least three over an infinite field $\mathbb{F}$. Then there exists $f \in R_1$ such that $R/P + (f)$ is also a domain.*

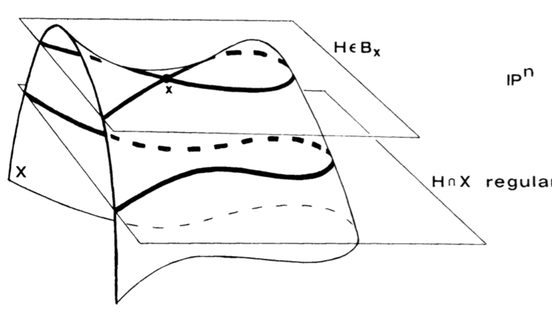

Figure 3. An illustration of Bertini's theorem.

[1] a technical condition which insures that if $\dim(R/P) = d$ then there is a sequence $f_1, \ldots, f_d \in R_+$ such that for $1 \leq i \leq d$, f_i is a non zero-divisor on $R/P + (f_1, \ldots, f_{i-1})$.

Corollary 11 (Stanley [7])**.** *Let R/P be a Cohen-Macaulay graded domain of dimension greater or equal than two. Then the h-vector of R/P is also the h-vector of a Cohen-Macaulay graded domain of dimension two (that is, the homogeneous coordinate ring of an irreducible curve).*

A further step after using Bertini's theorem would be to further intersect the curve from Corollary 11 with a line ending up with a set of points. After slicing by a general enough line, we get a set of points Γ such that all subsets of Γ of the same size have the same Hilbert function. This property is called the **uniform position** property (UPP).

Theorem 12 (Harris [2])**.** *Let P be a prime ideal such that the Krull dimension of R/P is 2. Then there exists $f \in R_1$ such that $V(P + (f))$ is a (reduced) set Γ of d points such that for every subset $\Gamma' \subseteq \Gamma$ of d' points and for every $i \geq 0$ we have*

$$H_{I_\Gamma(i)} = \min\{d', H_{I'_\Gamma(i)}\}.$$

Example. Six points of a conic in $\mathbb{P}^2$ are the vanishing set of a complete intersection ideal generated by a degree 2 equation (defining a conic) and a degree 3 equation (defining a cubic). Only the conic is pictured below. This could be irreducible as pictured in the first case or a union of two lines as in the last two cases.

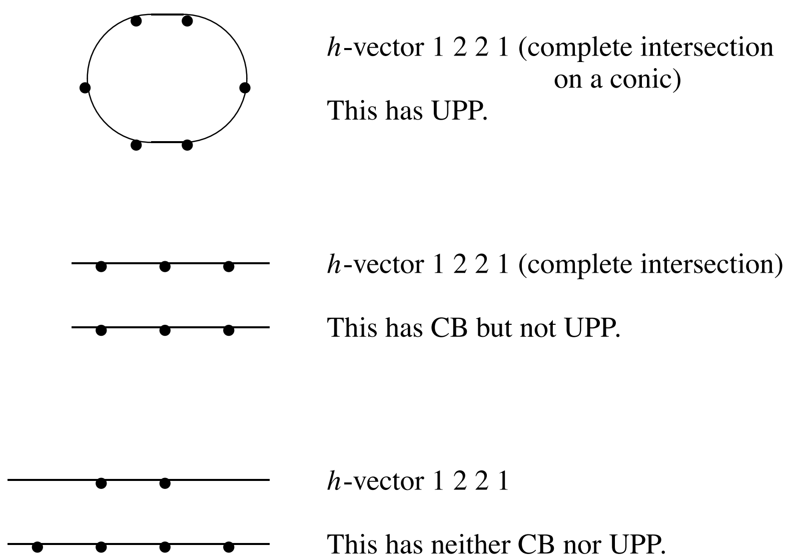

FIGURE 4. Six points on a conic in $\mathbb{P}^2$ and the UPP.

Question 13 (Reformulation of Question 9)**.** *What are all the possible Hilbert functions of points in $\mathbb{P}^n$ satisfying the UPP?*

There is a partial answer in the case $n = 2$:

Theorem 14 ([4])**.** *A finite sequence of natural numbers is the h-vector of R/I, where $V(I)$ is a set of points in $\mathbb{P}^2$ satisfying UPP if and only if $h_0 = 1, h_1 = 2$ and the h-vector of is admissible and of decreasing type, meaning that if $h_{i+1} < h_i$ then $h_{j+1} < h_j$ for all $j \geq i$.*

The moral of this section is that one can often reduce (in the Cohen-Macaulay case) the computation of the Hilbert function of a high-dimensional graded module to that of a module of Krull dimension 1 (or 0). These cases, which correspond to ideals defining (fat) points in $\mathbb{P}^n$ or Artinian algebras are thus particularly important.

4. Open questions

There are many open questions regarding Hilbert functions. I list some that are closest to my interests.

Conjecture 15 (The Hilbert function of a generic algebra [1])**.** *Let $F_1, \dots, F_r$ be homogeneous polynomials of degrees $d_1, \dots, d_r \geq 1$ in a polynomial ring $R = F[x_1, \dots, x_n]$. If $F_1, \dots, F_r$ are chosen "randomly" and $I = (F_1, \dots, F_r)$ then*

$$HS_{R/I}(t) = \frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1-t)^n}.$$

Conjecture 16 (Stanley's unimodality conjecture [7])**.** *The h-vector of a graded Cohen-Macaulay domain is* **unimodal***, i.e. there exists $0 \leq j \leq s$ such that*

$$h_0 \leq h_1 \leq h_2 \ldots \leq h_j \geq \ldots \geq h_{s-1} \geq h_s$$

.

Question 17 (Harris [2])**.** *What are the possible Hilbert functions of points in $\mathbb{P}^n$, $n \geq 4$ satisfying the UPP?*

An ideal defining a set of **fat points** is an ideal of the form

$$I = I_{p_1}^{m_1} \cap I_{p_2}^{m_2} \cap \cdots \cap I_{p_r}^{m_r}$$

where I_{p_i} is the ideal defining a point $p_i \in \mathbb{P}^n$.

The following conjecture states that any hypersurface vanishing at points $p_1, \dots, p_r \in \mathbb{P}^n$ with to order $m_1, \dots, m_r$ respectively must have degree $d \geq \frac{m_1+m_2+\cdots+m_r}{\sqrt{n}}$.

Conjecture 18 (Nagata's conjecture [5])**.** *If $I = I_{p_1}^{m_1} \cap I_{p_2}^{m_2} \cap \cdots \cap I_{p_r}^{m_r}$ is an ideal defining r fat points in $\mathbb{P}^n$ and $d > 0$ is an integer such that $H_I(d) > 0$ then*

$$\sqrt{n} \cdot d \geq m_1 + m_2 + \cdots + m_r.$$

References

[1] Fröberg, *An inequality for Hilbert series of graded algebras.* Math. Scand. 56 (1985), no. 2, 117–144.
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[3] F. S. Macaulay, *Some properties of enumeration in the theory of modular systems*, Proc. London Math. Soc., 26 (1927), 531–555.
[4] R. Maggioni, A. Ragusa, *The Hilbert function of generic plane sections of curves of* $\mathbb{P}^3$. Invent. Math. 91 (1988), no. 2, 253–258.
[5] M. Nagata, *On the* 14*-th problem of Hilbert*, Am. J. Math., 81 (1959), 766–772.
[6] R. P. Stanley, *Hilbert functions of graded algebras*, Advances in Math. 28 (1978), no. 1, 57–83.
[7] R. P. Stanley, *On the Hilbert function of a graded Cohen-Macaulay domain.* J. Pure Appl. Algebra 73 (1991), no. 3, 307–314.

Exercises on Hilbert functions

(1) (a) Prove that for $R = \mathbb{F}[x_1, \ldots, x_n]$, the Hilbert function is $H_R(i) = \binom{n+i-1}{i}$ using a combinatorial argument.
(b) Prove that for $R = \mathbb{F}[x_1, \ldots, x_n]$, the Hilbert function is $H_R(i) = \binom{n+i-1}{i}$ and the Hilbert series is $HS_R(t) = \frac{1}{(1-t)^n}$ by induction on n.

(2) Prove that if $R = \mathbb{F}[x_1, \ldots, x_n]$ and $f_1, \ldots, f_d \in R_+$ are such that for $1 \leq i \leq d$, f_i is a non zero-divisor on $R/(f_1, \ldots, f_{i-1})$, then

$$HS_{R/I}(t) = \frac{\prod_{i=1}^{s}(1 - t^{d_i})}{(1-t)^n}.$$

(3) Prove that for $R = \mathbb{F}[x, y, z]$ and $I = (F, G)$ such that $\deg(F) = 2, \deg(G) = 3$ and $\gcd(F, G) = 1$ the h-vector of R/I is $1, 2, 2, 1$.

(4) (a) Prove that a set of six points in $\mathbb{P}^2$ that lie on two lines does not satisfy the Uniform Position Property.
(b) Prove that a set of six points in $\mathbb{P}^2$ that lie on an irreducible conic satisfies the Uniform Position Property.

(5) Does there exist a set of points in $\mathbb{P}^3$ having the Uniform Position Property and h-vector $1, 3, 6, 5, 6$?

(6) Prove that the h-vector of a Cohen-Macaulay graded domain of dimension greater or equal to two is also the h-vector of a graded domain of dimension two using Bertini's Theorem.

(7) Let I be a homogeneous ideal of $R = \mathbb{F}[x_1, \ldots, x_n]$ and $m = (x_1, \ldots, x_n)$. The *fiber ring* of I is $\mathcal{F}(I) = \bigoplus_{i \geq 0} I^i/mI^i$. Show that $\mathcal{F}(I)$ is an $\mathbb{F}$-algebra and find what the Hilbert function of $\mathcal{F}(I)$ counts.

(8) A graded finite dimensional $\mathbb{F}$-algebra A is called Gorenstein provided that
- $A = A_0 \oplus A_1 \oplus \cdots \oplus A_s$ with $A_s \cong \mathbb{F}$ and
- for any $0 \leq i \leq s$ and $a \in A_i$ there is $a' \in A_{s-i}$ such that $aa' \neq 0$.

Prove that h-vectors of Gorenstein algebras are symmetric ($h_i = h_{s-i}$) using the following outline:

Let $R = \mathbb{F}[x_1, \ldots, x_n]$, $A = R/I$ and $J = 0 :_A I$.

(a) show that $J_{s-i} = \ker(A_{s-i} \to \mathrm{Hom}(I_i, A_s))$;
(b) show that there is an injective map $J_{s-i} \to \mathrm{Hom}(A/I_i, A_s)$;
(c) deduce that $H_A(s-i) \leq H_I(i) + H_J(s-i) \leq H_A(i)$ for all $0 \leq i \leq s$;
(d) conclude that $H_A(s-i) = H_A(i)$ for all $0 \leq i \leq s$.