Rees Algebras of monomial ideals and modules

Kuei-Nuan Lin

Penn State University, Greater Allegheny

kul20@psu.edu

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Definition 1. Throughout this note, $R = k[x_1, ..., x_n]$ will be a polynomial ring over a field k and $I \subseteq R$ a monomial ideal of R generated by some polynomials of the same degree. Let $G(I) = \{f_1, ..., f_u\}$ be a minimal generating set of I. Recall that the *Rees algebra* of I is defined to be

$$\mathcal{R}(I) = \oplus I^n \cong R[It] = \oplus I^n t^n \subseteq R[t].$$

Let $\varphi : R[T_1, ..., T_u] \to R[It]$ be the surjective homomorphism defined by $\varphi(T_i) = f_i t$. Then,

$$R[T_1, ..., T_u]/\ker\varphi \cong R[It].$$

The ideal $\ker\varphi$ will be called as the *defining ideal of Rees algebra* of I, and also be called as the *Rees ideal* of I. Notice that $\ker\varphi$ is a prime ideal such that $\text{height}(\ker\varphi) = u - 1$.

Once you find the generators of the ideal $\ker\varphi$ by hand, you can use the command `reesIdeal` of the software `Macaulay2` to check your answer.

Example 2. Let $I = (x_1, x_2)$. The defining ideal of $\mathcal{R}(I)$ is generated by the irreducible element $T_1x_2 - T_2x_1$ because $\varphi(T_1x_2 - T_2x_1) = x_1x_2 - x_2x_1 = 0$ and the height of the principal prime ideal $(T_1x_2 - T_2x_1)$ is 1.

Exercise 3. Let $I = (x_1^2, x_1x_2, x_2^2)$. Find the defining ideal of $\mathcal{R}(I)$.

Exercise 4. Let $I = (x_1x_2, x_2x_3, x_3x_1)$. Find the defining ideal of $\mathcal{R}(I)$.

Question: What can you say about the biggest differences between the Rees ideals of the two exercises above?

Definition 5. Let I be a square-free monomial ideal generated by degree 2 elements. We can define an associated simple graph D with the vertex set $V = \{1, ..., n\}$ and the edge set $E(D) = \{\{i, j\} \mid x_i x_j \in G(I)\}$.

Example 6. If $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_1)$, then $E(D) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$.

Exercise 7. Draw the graph defined by the ideal $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1, x_2x_5, x_3x_6)$.

Theorem 8 (Villarreal)**.** *Let I be a square-free monomial ideal generated by degree 2 elements. Then the* $\ker\varphi$ *is generated by the first syzygy and the even closed walk of D.*

Example 9. In Example 6, the Rees ideal is

$$\ker\varphi = (x_3T_{12} - x_1T_{23}, x_4T_{23} - x_2T_{34}, x_1T_{34} - x_3T_{41}, x_4T_{12} - x_2T_{41}, T_{12}T_{34} - T_{23}T_{41}).$$

Exercise 10. Find the generators of $\ker\varphi$ where

$$I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1, x_2x_5, x_3x_6).$$

Exercise 11. Find the generators of $\ker\varphi$ where

$$I = (x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_5x_6, x_6x_7, x_7x_8, x_8x_5).$$

Definition 12. Let $I_1, ..., I_l$ be ideals of R. Then, $M = I_1 \oplus \cdots \oplus I_l$ is the *ideal module* defined by $I_1, ..., I_l$. We can define the *multi-Rees algebra* of $I_1 \oplus \cdots \oplus I_l$ as

$$\mathcal{R}(M) = \mathcal{R}(I_1 \oplus \cdots \oplus I_l) = \oplus I_1^{a_1} \cdots I_l^{a_l} t_1^{a_1} \cdots t_l^{a_l} \subseteq R[t_1, t_2, \cdots, t_l]$$

and consider the homomorphism

$$\varphi : R[\{T_{i,j}\}] \to \mathcal{R}(I_1 \oplus \cdots \oplus I_l)$$

via $\varphi(T_{i,j}) = f_{i,j}t_i$ if $G(I_i) = (\{f_{i,j}\})$. The kernel, $\ker\varphi$, is called the *defining ideal of the Rees algebra* of M.

Example 13. Let $M = (x^2, xy, y^2) \oplus (x^2, xy, y^2)$ be a module in $R[x, y]$. In `Macaulay2`, you can input a matrix to define a module. You can also use the `reesIdeal` to obtain the generators of the defining ideal of the Rees algebra of the module.

```
i1 : R=QQ[x,y];

i2 : m=matrix{{x^2,x*y,y^2,0,0,0},{0,0,0,x^2,x*y,y^2}};

             2       6
o2 : Matrix R  <--- R

i3 : M=image m;

i4 : K=reesIdeal M;

o4 : Ideal of R[w , w , w , w , w , w ]
                 0   1   2   3   4   5

i5 : transpose gens K

o5 = {-1, -3} | yw_4-xw_5     |
     {-1, -3} | yw_3-xw_4     |
     {-1, -3} | yw_1-xw_2     |
     {-1, -3} | yw_0-xw_1     |
     {-2, -4} | w_4^2-w_3w_5  |
     {-2, -4} | w_2w_4-w_1w_5 |
     {-2, -4} | w_1w_4-w_0w_5 |
     {-2, -4} | w_2w_3-w_0w_5 |
     {-2, -4} | w_1w_3-w_0w_4 |
     {-2, -4} | w_1^2-w_0w_2  |

                                     10                                  1
o5 : Matrix (R[w , w , w , w , w , w ])   <--- (R[w , w , w , w , w , w ])
                0   1   2   3   4   5               0   1   2   3   4   5
```

Consider instead the matrix,

$$N = \begin{bmatrix} x & w_0 & w_1 & w_3 & w_4 \\ y & w_1 & w_2 & w_4 & w_5 \end{bmatrix}$$

and the associated module L. Then, one can check easily that the Rees ideal of L is generated by the 2 by 2 minors of the matrix N.

Exercise 14. Let $M = (x^3, x^2y, xy^2, y^3) \oplus (x^2, xy, y^2)$. Find the generators of the Rees ideal by hand. Then, check your answer using `Macaulay2`. With respect to this M, find a matrix N such that the Rees ideal is generated by the 2 by 2 minors of the matrix N.

Exercise 15. Let $M = (x^4, x^3y, x^2y^2, xy^3, y^4) \oplus (x^3, x^2y, xy^2, y^3) \oplus (x^2, xy, y^2)$. Find the generators of the Rees ideal by hand. Then, check your answer using `Macaulay2`. With respect to this M, find a matrix N such that the Rees ideal is generated by the 2 by 2 minors of the matrix N.